NSC 123
DIFFERENTIAL EQUATIONS
SPRING 2018

Quiz 1 Solutions

(1) Write a differential equation that fits the physical description: "The velocity at time t of a cat moving along a straight line is proportional to the fourth power of its position x."

$\frac{dx}{dt} = a \cdot x^4$

(2) Determine for which values of m the function $y = e^{mx}$ is a solution to the given equation.

(a) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 0$

Letting $y = e^{mx}$ we have that $\frac{dy}{dx} = me^{mx}$ and $\frac{d^2y}{dx^2} = m^2e^{mx}$. Substituting this into our given equation, we obtain

$$m^2e^{mx} + 6me^{mx} + 5e^{mx} = 0.$$

Factoring out e^{mx} from each term, we arrive at $e^{mx}(m^2 + 6m + 5) = 0$. So we have that either e^{mx} is 0, which is impossible, or $m^2 + 6m + 5$ is. Thus

$$m^2 + 6m + 5 = 0.$$

To solve for m, we can factor the expression to obtain $(m + 1)(m + 5) = 0$, which means that m is either -1 or -5 for there to be a solution to the given equation of the form $y = e^{mx}$.

(b) $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$

Again, letting $y = e^{mx}$ we have that $\frac{dy}{dx} = me^{mx}$, $\frac{d^2y}{dx^2} = m^2e^{mx}$, and $\frac{d^3y}{dx^3} = m^3e^{mx}$. Substituting this into our given equation, we obtain

$$m^3e^{mx} + 3m^2e^{mx} + 2m^2e^{mx} = 0.$$

Factoring out me^{mx} from each term, we arrive at $me^{mx}(m^2 + 3m + 2) = 0$. So we have that either me^{mx} is 0, which means that $m = 0$, or

$$m^2 + 3m + 2 = 0.$$

We can factor the last expression to obtain $(m + 1)(m + 2) = 0$, which means that m is either 0, -1, or -2 for there to be a solution to the given equation of the form $y = e^{mx}$.

(3) Solve the equation for a general solution, and describe the general solution graphically:

$$\frac{dy}{dx} = -\frac{x}{y}$$

This is a separable equation, and to solve it we need to solve

$$\int y\,dy = -\int x\,dx$$

which gives us $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + c$. This gives us an equation of the form $x^2 + y^2 = 2c$; the equation of a circle of radius $\sqrt{2c}$ centered at $(0,0)$.

(4) Solve the initial value problem:

$$\frac{dy}{d\theta} = \sin\theta + y^2\sin\theta$$
$$y(0) = \sqrt{3}$$

First let's notice that we can rearrange the equation a little bit to easily see that it is separable:

$$\frac{dy}{d\theta} = \sin\theta(1+y^2).$$

By rearranging, we obtain:

$$\int \frac{1}{1+y^2}\,dy = \int \sin\theta\,d\theta.$$

If you don't remember how to integrate the left hand side, what you can do is a trig substitution. I tend to remember very few trig identities, but I do remember $\sin^2 u + \cos^2 u = 1$. Except I want to substitute something for y. So I can divide the trig identity by $\cos^2 u$ to get the trig identity

$$1 + \tan^2 u = \sec^2 u.$$

So let $y = \tan u$. Then $\frac{dy}{du} = \sec^2 u$. If you don't remember this, just use the quotient rule, because you should remember the derivative of sine and cosine. Ultimately this means that we have reduced the above integral to the following:

$$\int du = \int \sin\theta\,d\theta,$$

giving us $\tan^{-1} y = -\cos\theta + c$. Using our initial conditions, we get that

$$\tan^{-1}\sqrt{3} = -\cos 0 + c,$$

ie., $c = \frac{\pi}{3} + 1$. This gives us the solution:

$$y = \tan\left(\frac{\pi}{3} + 1 - \cos\theta\right).$$

(5) Suppose a brine containing 0.3 kilograms of salt per liter runs into a tank initially filled with 400 liters of water containing 2 kilograms of salt. If the brine enters at a rate of 10 liters per minute, the mixture is kept uniform by stirring, and the mixture flows out at the same rate, find a formula giving the mass of salt in the tank after t minutes.

Let s be the amount of salt, in kilograms, in the mixture at time t minutes. We know that the rate of change in the amount of salt in the mixture at time t is going to be equal to the rate of change of the salt quantity coming in, minus the rate of change of the salt quantity coming out. So we can write the following. (I will include units so you see somewhat more clearly perhaps why it works out this way.)

$$\frac{ds}{dt} = \frac{0.3\,kg}{L} * \frac{10\text{ L}}{\text{min}} \;-\; s\,kg * \frac{1}{400\,L} * \frac{10\text{ L}}{\text{min}},$$

So our equation comes out to:

$$\frac{ds}{dt} = 3 - \frac{s}{40}.$$

This is a separable equation. To solve this we need to solve:

$$40 \int \frac{1}{120 - s}\,ds = \int dt.$$

To solve this of course we use a simple "u substitution". Let $u = 120 - s$ so that $\frac{du}{ds} = -1$. This gives us:

$$-40 \ln|120 - s| = t + c.$$

Rearranging this equation a bit, we get

$$s = 120 - Ce^{-\frac{t}{40}}$$

Of course here C is different than c. To find C, we need to plug in the initial condition, namely that there is $2\,kg$ of salt originally - at time $t = 0$. So $C = 118$. So finally we get that our equation giving the mass of salt s in the tank after t minutes is:

$$s = 120 - 118e^{-\frac{t}{40}}.$$