NSC 123
DIFFERENTIAL EQUATIONS
SPRING 2018

Quiz 2 Solutions

(1) Classify each differential equation as exact, linear, separable, or none of these. Name all appropriate classifications, if any.

(a) $y\,dy + x\,dx = 0$

Exact? Yes. $\frac{\partial}{\partial x}y = 0 = \frac{\partial}{\partial y}x$.
Linear? No.
Separable? Yes. Rewrite as $y\,dy = -x\,dx$.

(b) $y^2\,dx + (2xy + \cos y)\,dy = 0$

Exact? Yes. $\frac{\partial}{\partial y}[y^2] = 2y = 2y = \frac{\partial}{\partial x}[2xy + \cos y]$.
Linear? Yes. Rewrite as $\frac{dx}{dy} + \frac{2x}{y} = \frac{\cos y}{y}$.
Separable? No.

(c) $1/y\,dx - (3y - x/y^2)\,dy = 0$

Exact? No. $\frac{\partial}{\partial y}[1/y] = -1/y^2 \neq 1/y^2 = \frac{\partial}{\partial x}[-(3y - x/y^2)]$.
Linear? Yes. Rewrite as $\frac{dx}{dy} + \frac{x}{y} = 3y^2$.
Separable? No.

(d) $(x^2y + x^4\cos x)\,dx - x^3\,dy = 0$

Exact? No.
Linear? Yes. Rewrite as $\frac{dy}{dx} - \frac{y}{x} = -x\cos x$.
Separable? No.

(2) Solve the initial value problem.

$$\frac{dy}{dx} - \frac{y}{x} = xe^x,$$
$$y(1) = e + 1$$

This is linear. To solve, we need to first find the integrating factor $\mu(x)$.

$$\mu(x) = e^{\int -\frac{1}{x}\ dx} = e^{-\ln|x|} = \frac{1}{e^{\ln|x|}} = \frac{1}{x}.$$

Thus

$$y(x) = \frac{1}{\mu}\int \mu(x)xe^x\ dx = x\int e^x\ dx = x(e^x + c).$$

To solve for c we use the initial condition.

$$e + 1 = y(1) = e + c \implies c = 1.$$

So $y = xe^x + x$.

(3) Solve the initial value problem.

$$(ye^{xy} - 1/y)\,dx + (xe^{xy} + x/y^2)\,dy = 0,$$
$$y(1) = 1$$

First we should check for exactness. We have that

$$\frac{\partial}{\partial y}\left[ye^{xy} - 1/y\right] = e^{xy} + yxe^{xy} + 1/y^2$$

and

$$\frac{\partial}{\partial x}\left[xe^{xy} + x/y^2\right] = e^{xy} + yxe^{xy} + 1/y^2$$

So the equation is exact. So there is an $F(x, y)$ such that $\frac{\partial F}{\partial x} = ye^{xy} - 1/y$ and $\frac{\partial F}{\partial y} = xe^{xy} + x/y^2$. To satisfy the first equation, it must be that for some $g(y)$ we have:

$$F(x, y) = \int (ye^{xy} - 1/y)\ dx + g(y) = e^{xy} - x/y + g(y).$$

So to satisfy the second equation, we must have that:

$$xe^{xy} + x/y^2 = \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}[e^{xy} - x/y + g(y)] = xe^{xy} + x/y^2 + g'(y).$$

Thus $g'(y) = 0$. So our equation is $C = e^{xy} - x/y$.
To solve for C, plug in the initial condition: $C = e - 1$. Thus we have

$$e - 1 = e^{xy} - x/y.$$

(4) A chocolate mixture flows at a constant rate of 6 liters per minute into a large tank that initially held 200 L of a mixture that is 10% chocolate. The mixture inside the tank is kept well stirred, and flows out of the tank at a rate of 8 liters per minute. If the mixture entering the tank is 50% chocolate, determine the volume of chocolate in the tank after t minutes. When will chocolate consist of a quarter of the total volume in the tank?

Let $V(t)$ be the volume in liters of chocolate at time t. Then:

$$\frac{dV}{dt} = \frac{1}{2} * 6\frac{L}{min} - \frac{V(t)}{200 - 2t} * 8\frac{L}{min}$$

Without units, this gives us:

$$\frac{dV}{dt} = 3 - \frac{4V}{100 - t}.$$

Notice that this is linear:

$$\frac{dV}{dt} + \frac{4V}{100 - t} = 3.$$

So to solve, first we need to find the integrating factor $\mu(t)$

$$\mu(t) = e^{\int \frac{4}{100-t} dt} = e^{-4 \ln|100-t|} = (e^{\ln|100-t|})^{-4} = (100-t)^{-4}.$$

So we have our equation for V(t):

$$V(t) = (100-t)^4 \int 3(100-t)^{-4}\ dt = (100-t)^4[(100-t)^{-3} + c] = (100-t) + c(100-t)^4.$$

To solve for t, we use the initial condition, which states that there are 20 liters of chocolate initially.

$$20 = 100 + c(100)^4 \implies c = \frac{-80}{10^8} = -8 \times 10^{-7}.$$

So the volume of chocolate in the tank after t minutes is given by the equation

$$V(t) = (100-t) - \frac{8(100-t)^4}{10^7} = (100-t)\left(1 - \frac{8(100-t)^3}{10^7}\right).$$

To answer the second part of this question, we've already had to figure out that the total volume of the mixture is given by $200 - 2t$. We want to know when a quarter of this is chocolate. So we want to solve for t in the following equation:

$$\frac{1}{4}(200-2t) = \frac{1}{2}(100-t) = V(t).$$

Thus we need to solve for t in the following:

$$\frac{1}{2} = 1 - \frac{8(100-t)^3}{10^7}.$$

Doing some more algebra, we see that

$$t = 100 - \sqrt[3]{\frac{10^7}{16}} \approx 14.5.$$

Thus chocolate will consist of a quarter of the total volume in approximately 14 minutes and 30 seconds.