NSC 123
DIFFERENTIAL EQUATIONS
SPRING 2018

Quiz 3 Solutions

(1) Suppose that at 10pm the outside temperature, $M(t)$, stays constant at 20°F, there is no additional heating rate, $H(t)$, in the shack outside the science building, and the furnace/air conditioner rate $U(t)$ is 0. Determine $T(t)$, the temperature inside the shack outside the science building at time t, given that at 10pm the temperature inside the shack outside the science building is 40°F.
Hint: Recall that in general,

$$\frac{dT}{dt} = K[M(t) - T(t)] + H(t) - U(t)$$

where K is a positive constant depending on the physical properties of the building, and should be left as a constant in this problem.
Here we are setting $\mathtt{T}(0) = 40°\mathtt{C}$ for simplicity, so we are saying t is the amount of time elapsed since 10pm. We arrive at the equation $\frac{dT}{dt} = K(20 - t)$. This is linear, in standard form:

$$\frac{dT}{dt} + KT = 20K.$$

So the solution is

$$T = e^{-Kt}\left[\int 20Ke^{Kt}\ dt\right] = e^{-Kt}\left[20e^{Kt} + C\right] = 20 + Ce^{-Kt}.$$

Our initial condition tells us that $40 = T(0) = 20 + C$ thus $C = 20$. Thus

$$T(t) = 20(1 + e^{-Kt})$$

is the equation giving the temperature inside the shack outside the science building at time t.

(2) Assume that a gummy bear with mass m is given an initial downward velocity v_0 and allowed to fall under the influence of gravity. Assuming the gravitational force is constant and the force due to air resistance is proportional to the velocity of the gummy bear, determine the equation of motion, $x(t)$, for the gummy bear.
Hint: In this situation, applying Newton's Law of Motion, the differential equation described is given by

$$m\frac{dv}{dt} = mg - bv$$
$$v(0) = v_0$$

where $v(t)$ is the velocity of the gummy bear at time t, b is the proportionality constant of the force due to air resistance and g is the gravitational force.
The book derives this. Should get, assuming the initial position $x(0) = 0$,

$$x(t) = \frac{mg}{b}t + \frac{m}{b}\left(v_0 - \frac{mg}{b}\right)(1 - e^{-bt/m}).$$

(3) Find a general solution to the following equation: $3z' + 11z = 0$.
Here the auxiliary equation is $3r + 11 = 0$, obtained by trying to let $z = e^{rt}$. This just yields one solution, $r = -11/3$. Thus

$$y = Ce^{-11t/3}$$

is our general solution.

(4) Solve the initial value problem:

$$y'' - 2y' + 2y = 0$$
$$y(\pi) = e^{\pi}$$
$$y'(\pi) = 0$$

Here the auxiliary equation is $r^2 - 2r + 2 = 0$. The roots of this are $r = 1 \pm i$. Thus

$$y(t) = C_1 e^{(1+i)t} + C_2 e^{(1-i)t}$$

where C_1 and C_2 are constants to be determined. To find them, use the initial conditions. First of all,

$$e^{\pi} = y(\pi) = C_1 e^{\pi}(\cos\pi + i\sin\pi) + C_2 e^{\pi}(\cos\pi - i\sin\pi),$$

and dividing by $-e^{\pi}$ on both sides, yields

$$-1 = C_1 + C_2.$$

Secondly,

$$y'(t) = C_1(1+i)e^{(1+i)t} + C_2(1+i)e^{(1-i)t}.$$

So plugging in the initial condition, we obtain

$$\begin{aligned} 0 = y'(\pi) &= C_1(1+i)e^{(1+i)\pi} + C_2(1+i)e^{(1-i)\pi}. \\ &= C_1(1+i)[\cos\pi + i\sin\pi] + C_2(1-i)[\cos\pi - i\sin\pi] \\ &= C_1(1+i)[-1] + C_2(1-i)[-1] \\ &= -(1+i)C_1 + (1-i)C_2. \end{aligned}$$

Using algebra, we obtain $C_1 = \frac{1-i}{2i}$ and $C_2 = -\frac{(1+i)}{2i}$. Finally, again using Euler's formula,

$$\begin{aligned} y(t) &= e^t\left[\frac{1-i}{2i}(\cos t + i\sin t) - \frac{1+i}{2i}(\cos t - i\sin t)\right] \\ &= e^t\left[\sin t - \cos t\right]. \end{aligned}$$