NSC 123
DIFFERENTIAL EQUATIONS
SPRING 2018

Quiz 4 Solutions

(1) Find a particular solution to the differential equation $\theta''(t) - \theta(t) = t \sin t$.
This is §4.4 # 16. To solve, guess the following particular solution (given here with its derivatives):

$$\theta_p(t) = (At + B)\sin t + (Ct + D)\cos t$$
$$\theta_p'(t) = (A - D - Ct)\sin t + (At + B + C)\cos t$$
$$\theta_p''(t) = (-At - B - 2C)\sin t + (2A - D - Ct)\cos t.$$

Assuming this is a solution, we have that

$$t \sin t = \theta_p''(t) - \theta_p(t) = (-2At - 2B - 2C)\sin t + (2A - 2D - 2Ct)\cos t.$$

This means that $C = 0, B = 0, A = -1/2, D = 1/2$. Therefore

$$\theta_p(t) = \frac{1}{2}(\cos t - t \sin t).$$

(2) Find a general solution to the differential equation $y'' - 2y' + y = t^{-1}e^t$.
This is §4.6 # 5. First let's find the particular solution to the associated homogeneous equation. Our associated auxiliary equation is $r^2 - 2r + 1 = (r-1)^2$. Since we have a double root $r = 1$, this means we have as our solution

$$y_h(t) = C_1 e^t + C_2 t e^t.$$

Let's now find the particular solution using variation of parameters. Here

$$y_p(t) = v_1(t)e^t + v_2(t)te^t,$$

where $y_1(t) = e^t$ and $y_2(t) = te^t$ and we need to find v_1, v_2 satisfying:

$$v_1' y_1 + v_2' y_2 = 0$$
$$v_2' y_2' + v_1' y_1' = t^{-1}e^t.$$

Here $y_1'(t) = e^t$ and $y_2'(t) = te^t + e^t = e^t(t+1)$. Therefore we need to solve the following for v_1' and v_2':

$$v_1' e^t + v_2' t e^t = 0$$
$$v_1' e^t + v_2' e^t (t+1) = e^t t^{-1}.$$

Obtain $v_1' = -t v_2'$ from the first equation and substitute this into the second equation, dividing through by e^t, and we get that $v_1' = -1$ and $v_2' = t^{-1}$. Therefore $v_1(t) = -t$ and $v_2(t) = \ln|t|$. So our particular solution is

$$y_p(t) = -te^t + te^t \ln|t|.$$

Adding the homogeneous solution and the particular solution together, we obtain

$$y(t) = C_1 e^t + C_2 t e^t + t e^t \ln|t|.$$

(3) Two large tanks, each holding 20 L of liquid, are interconnected by pipes, with the liquid flowing from tank A into tank B at a rate of 4 L/min and from B into A at a rate of 2 L/min. The liquid inside each tank is kept well stirred. Simple syrup with a concentration of 0.5 kg/L of sugar flows into tank A at a rate of 4 L/min. The (diluted) simple syrup flows out of the system from tank A at 2 L/min and from tank B also at 2 L/min. If, initially, tank A contains pure water and tank B contains 2 kg of sugar, determine the mass of sugar in each tank at time $t \geq 0$.

Using x for the mass of sugar in tank A and y for the mass of sugar in tank B, should arrive at the following system:

$$x' = 2 + \frac{y}{10} - \frac{3x}{10}$$
$$y' = \frac{x}{5} - \frac{y}{5}.$$

Writing this in operator notation, this is

$$(5D+1)[y] - x = 0$$
$$(10D+3)[x] - y = 20.$$

Solving for x in the first equation, yielding $x = (5D+1)[y]$, and substituting into the second, we arrive at the following differential equation for y:

$$(50D^2 + 25D + 2)[y] = 20.$$

First let's find the homogeneous solution. The roots of the corresponding auxiliary equation are $r_1 = -1/10$, $r_2 = -2/5$. This yields

$$y_h(t) = C_1 e^{-t/10} + C_2 e^{-2t/5}.$$

It is not hard to see that using undetermined cofficients, the particular solution is $y_p(t) = 10$, and thus the general solution then comes out to be

$$y(t) = C_1 e^{-t/10} + C_2 e^{-2t/5} + 10.$$

Using the fact that $x = (5D+1)[y]$, this gives us

$$x(t) = C_1/2 \cdot e^{-t/10} - C_2 e^{-2t/5} + 10.$$

We plug in the initial conditions to solve for C_1 and C_2. This yields $C_1 = -12$ and $C_2 = 4$. So our final equations for the amount of sugar in each tank at time t are given by

$$y(t) = -12e^{-t/10} + 4e^{-2t/5} + 10$$
$$x(t) = -6e^{-t/10} - 4e^{-2t/5} + 10.$$