NOTES ON FOURIER TRANSFORMS

Christopher R. Tripp
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*In these notes we first review some relevant material from classical mechanics concerning oscillations and waves before introducing the Fourier series and the continuous Fourier transform. We then introduce the Discrete Fourier Transform (DFT) and the Fast Fourier Transform (FFT).

1 Oscillations and Simple Harmonic Motion

- Uniform circular motion is described by the following position vector:

$$\boldsymbol{r} = (r\cos\theta)\hat{\boldsymbol{\imath}} + (r\sin\theta)\hat{\boldsymbol{\jmath}} = (r\cos\omega t)\hat{\boldsymbol{\imath}} + (r\sin\omega t)\hat{\boldsymbol{\jmath}}$$

  If we project uniform circular motion onto one dimension, the result is called **simple harmonic motion**. Restoring forces make an equilibrium stable, and a small displacement from a stable equilibrium is what produces simple harmonic motion. Put another way, simple harmonic motion is any motion that is a combination of a sine and cosine of the form

$$x(t) = B_1\cos(\omega t) + B_2\sin(\omega t)$$

  This motion can be equivalently expressed as

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$

2 Traveling Waves

- A traveling **wave** is a disturbance traveling with a given **wave speed** v. The degree to which an element of the wave is disturbed at some position and time is called its **displacement from equilibrium**. The **amplitude** A of the wave is the maximum value of the displacement. A wave source that oscillates with simple harmonic motion will generate a sinusoidal wave, and the **frequency** f of the wave produced will match the frequency of the source's oscillations. The **period** T of the wave is the time interval for one cycle of the motion, and we note the following relationship:

$$T = \frac{1}{f}$$

- We can graph the displacement of one fixed point in space over time, and we will obtain a periodic sinusoidal graph. Alternatively, we can take a snapshot of a wave at a certain time, and graph the displacement of the whole wave at that fixed time. Sinusoidal waves are periodic in space as well as in time, so this graph in terms of position for a fixed time will also be periodic, where the distance spanned by one cycle of motion is called the **wavelength** λ of the wave.

  The period T of a wave is the time it takes for the disturbance at a fixed point in space to repeat itself. The wavelength λ of a wave is the distance it takes for the disturbance to repeat itself for a fixed instant in time. We see therefore that displacement is in fact a function of both position and time, and we write this as $D(x, t)$.

- If we analyze several history graphs of a wave over the span of one period, we will see that during a time interval of exactly one period T each crest of a sinusoidal wave travels forward a distance of exactly one wavelength λ. This is the fundamental relationship for sinusoidal waves. We therefore have $v = \lambda/T$, which is usually written as

$$v = \lambda f$$

3 The Mathematics of Sinusoidal Traveling Waves

- We can describe the displacement of a sinusoidal waves at a fixed time $t = 0$ as

$$D(x, t = 0) = A \sin\left(\frac{2\pi}{\lambda}x + \phi_0\right)$$

where ϕ_0 is the phase constant that characterizes the initial conditions. Now, we note that in a time interval t a wave will travel a distance of vt, we see that the displacement of a wave at time t. This means that whatever displacement the wave has at position x and time t is the same as the displacement the wave had at position $x - vt$ and time $t = 0$. In other words, $D(x, t) = D(x - vt, t = 0)$. We can substitute this into the previous equation as follows

$$\begin{aligned} D(x, t) &= A \sin\left(\frac{2\pi}{\lambda}(x - vt) + \phi_0\right) \\ &= A \sin\left(\frac{2\pi}{\lambda}x - \frac{2\pi v}{\lambda}t + \phi_0\right) \\ &= A \sin\left(kx - 2\pi f t + \phi_0\right) \\ &= A \sin\left(kx - \omega t + \phi_0\right) \end{aligned}$$

where we have defined the **wave number** $k = 2\pi/\lambda$, and recalling from simple harmonic motion that the angular frequency ω can be expressed as $\omega = 2\pi f$. This is the equation for a sinusoidal wave traveling in the positive x direction, and we can see that it is a function of both position x and time t.

4 Standing Waves

- If two sinusoidal traveling waves of the same wavelength, frequency, and amplitude coincide in space traveling in opposite directions, then by the principle of superposition they will combine to form one **standing wave** which does not travel but whose amplitude will oscillate up and down. The displacement of a standing wave is given by

$$D(x, t) = 2A \sin(kx) \cos(\omega t)$$

- If we consider a standing wave on a string tied down at both ends, we see that a standing wave on the string must have nodes at both ends. (Note that electromagnetic waves and sound waves in closed-closed tubes also require nodes at both ends.) Therefore, we have a boundary condition which imposes that the wavelength must be of the form $\lambda_m = 2L/m$ where L is the length of the string and m is a positive integer. Consequently, the frequency of the standing wave is also restricted to that of the form

$$f_m = \frac{v}{\lambda_m} = m\frac{v}{2L}$$

We see here that the lowest possible frequency is $f_1 = v/2L$, and this is referred to as the **fundamental frequency**. The other possible frequencies can be expressed as integer multiples of the fundamental frequency thusly: $f_m = mf_1$. These other possible frequencies are called **harmonics**, where f_2, for example, is called the "second harmonic." The possible standing waves are called the **modes** of the string. Note that each mode m has a unique wavelength and frequency.

5 Fourier Series

- If you hit a piano key, the resulting sound wave is periodic, and is composed of several sinusoidal waves with different frequencies, each called a "pure tone." Therefore, if we measure the pressure resulting from a sound wave as a function of x and t, then the pressure will be a sum of several sinusoidal functions, each representing one of the pure tones (that is, individual sinusoidal waves each with different frequencies). One of the pure tones will have a fundamental frequency, and the others will have corresponding harmonic frequencies. If $\sin(\omega t)$ and $\cos(\omega t)$ correspond to the fundamental frequency (in this case, $f_1 = \omega/2\pi$), then $\sin(n\omega t)$ and $\cos(n\omega t)$ correspond to the higher harmonics, where higher values of n result in higher frequencies and shorter periods. The overall net combination of the fundamental and the harmonics is a complicated periodic function with the period of the fundamental, and we can express this function as a sum of terms corresponding to the various harmonics; this sum is called a **Fourier series**.

  The exact same process is applicable for other types of waves besides sound. For example, we could describe the various proportions of different light frequencies in a given beam of light.

- In general, it turns out that *any* function with period T can be expressed as a linear combination of these sines and cosines as follows:

  $$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

  This is the Fourier series for $f(t)$. Constructing a waveform by adding together a fundamental frequency and harmonics of various amplitudes in a process called **Fourier synthesis**. The reverse process - determining the various frequencies and amplitudes present in a given waveform - is called **Fourier analysis**. The latter process is common in practice; for example, suppose we have already found a curve $f(t)$ experimentally and we now want to know the values of the amplitudes a_n and b_n for as many values of n as necessary. These amplitudes (that is, the Fourier coefficients) can be calculated as follows:

  $$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos(nx)\,dx$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin(nx)\,dx$$

- We can equivalently express the previous formula in terms of the complex exponential:

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n e^{in\omega t}$$

where the coefficients can be calculated as follows:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x)e^{-inx}\,dx$$

- A trick that is often used in evaluating trigonometric Fourier coefficients relies on the fact that cosine is an **even function** while sine is an **odd function**. In particular, since an even function has the same value at any point t as it does at point $-t$, we note that for even functions one can replace any integral from $-T$ to T by twice the integral from 0 to T.

6 Fourier Transforms

- Unlike when we hit a piano key, some other sound waves are *not* periodic. Describing these sounds waves (that is, graphing pressure as a function of time) therefore requires not just a fundamental frequency, its amplitude, and a set of discrete harmonics and their amplitudes (as was true for the case of the piano), but rather a continuous range of frequencies.

  In this way, a non-periodic function may be thought of as a limiting case of a periodic function, where the period tends to infinity, and consequently the fundamental frequency tends to zero. The harmonics are more and more closely spaced and in the limit there is a continuous range of harmonics, each of infinitesimal amplitude. Therefore, the summation of the Fourier series is replaced by integration, and the graph of our waveform is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\,d\alpha$$

  Here we see that for a given time x we graph sinusoids for a continuous range of frequencies α, where $g(\alpha)$ represents the amplitude for a given frequency. If we compare this to the complex exponential forms of the Fourier series, we note that the frequency α corresponds to n, the amplitude $g(\alpha)$ corresponds to c_n, and $\int_{-\infty}^{\infty}$ corresponds to $\sum_{-\infty}^{+\infty}$ since we are now graphing a continuous range of frequencies rather than a set of discrete frequencies. That is, the quantity α is a continuous analog of the integral-valued variable n, and so the set of coefficients c_n has become a function $g(\alpha)$.

  The **Fourier transform** of the previous equation is

$$g(\alpha) = \int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx$$

  We note that this represents amplitude as a function of frequency. For a given frequency α, the Fourier transform will graph its amplitude. So, if a certain frequency is present in the original signal, the Fourier transform will show a peak at that frequency. In this way the Fourier transform shows what frequencies are in a signal.

7 Discrete Fourier Transforms

- In order to make use of computational tools, we can discretize the continuous Fourier transform. If $F(x)$ is the original waveform (signal) at time x, then we can discretize it with N points as

$$F(x) = \sum_{j=-N/2+1}^{N/2} G(j) e^{2\pi i \frac{jx}{N}}$$

  and the **Discrete Fourier Transform** of that waveform is given by

$$G(j) = \frac{1}{N} \sum_{x=1}^{N} F(x) e^{-2\pi i \frac{jx}{N}}$$

  For a given frequency j, this last formula gives its amplitude $G(j)$ in the original signal.

- Because sinusoids are orthogonal at different frequencies, the sinusoidal terms in the DFT form an orthogonal basis of the space $\mathbb{C}^x$ (that is, the sinusoidal terms in the DFT are linearly independent and span the space $\mathbb{C}^x$) which can be normalized to obtain an orthonormal basis for $\mathbb{C}^x$. In this way we can see that the argument of the summation is the dot product of the original function with an orthogonal basis vector, thereby extracting the component of the original function along this new basis vector. We then sum over all the various values of x (that is, the components in the original basis), and we obtain one coefficient of the vector in the new basis.

  In other words, the Fourier transform defines the coefficients of the original function with respect to a new basis (complex sinusoids). It obtains these coefficients by projecting the original function onto each (orthogonal) basis function. In this sense it is a linear unitary transform. We see then that the DFT is proportional to the set of coefficients of projection onto the sinusoidal basis set.

- We note that the norm of the DFT sinusoids is $\sqrt{N}$. So elements of the orthonormal set are of the form

$$\frac{e^{2\pi i \frac{jx}{N}}}{\sqrt{N}}$$

- In computation, we can first define a function which takes a value of x and returns a corresponding basis vector. We then calculate a Fourier coefficient $G(j)$ for a given frequency j by summing the dot products of the original function and the various x-bases, with the latter determined by calling the function we previously defined. Finally, we plot these Fourier coefficients.

8 Fast Fourier Transforms

- Let us first recognize that the DFT is essentially matrix multiplication, insofar as it multiplies a column vector of size N (consisting of the various values of the original function) by a square matrix containing all the complex sinusoidal terms, which obtains a column vector consisting of the various Fourier coefficients. Since the process of matrix multiplication requires N^2 multiplications for its completion, these calculations can quickly become quite cumbersome for large data sets. **Fast Fourier Transforms** ameliorate this problem by reducing the number of multiplications to $2N \log_2(N)$. It essentially does this by factorizing the matrix of complex sinusoids.

- When N is a power of 2, the Cooley-Tukey algorithm is a recursion which at each stage puts the values of the input function $F(x)$ where the index x is even into one column vector of length $N/2$, and then puts the values of the input function $F(x)$ where the index x is odd into another column vector of length $N/2$. The DFT is then calculated for each of these two column vectors, and the results are combined to produce the DFT of the whole array of N inputs. This process is performed recursively, continuing until only 2×2 matrices are left, which have computationally trivial Fourier transforms. We can express this procedure as

$$G(j) = E(j) + e^{-2\pi i \frac{j}{N}} O(j)$$

$$G(j + \frac{N}{2}) = E(j) - e^{-2\pi i \frac{j}{N}} O(j)$$

where $E(j)$ and $O(j)$ are the even-indexed inputs and odd-indexed inputs, respectively.

References

[1] Anand, *A Brief Study of Discrete and Fast Fourier Transforms*

[2] Boas, *Mathematical Methods in the Physical Sciences*

[3] James, *A Student's Guide to Fourier Transforms*

[4] Knight, *Physics for Scientists and Engineers*

[5] Mahoney, *Fourier Notes*

[6] Taylor, *Classical Mechanics*

[7] Wikipedia, *Fourier Transform*

[8] Wikipedia, *Discrete Fourier Transform*