# Notes on Separation Logic Optics

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## 1 Terminology and Notation

- "Prosets" are preordered sets; "preorders" are relations. This is strictly analogous to "poset" and "partial order".
- $\Omega$ is the poset of truth values, with implication as the ordering.
- Because most of the prosets we are interested in are those that serve as algebras of assertions/propositions, we will typically write $P \vdash Q$ rather than $P \leq Q$.

## 2 Background on optics

Fix the following ingredients:

- Categories $\mathcal{C}$ and $\mathcal{D}$.
- A monoidal category $\mathcal{M}$.
- *Actions* of $\mathcal{M}$ on $\mathcal{C}$ and $\mathcal{D}$; i.e., strong monoidal functors $L : \mathcal{M} \to [\mathcal{C}, \mathcal{C}]$ and $R : \mathcal{M} \to [\mathcal{D}, \mathcal{D}]$, where the codomain functor categories have functor composition as their monoidal product. For $M \in \mathcal{M}, A \in \mathcal{C}$, we will typically denote $L(M)(A)$ by $M \cdot A$; we will also denote $R$ this way unless it is vital, and otherwise impossible, to distinguish $L$ and $R$.

One of the simplest examples of this set of ingredients is when $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{M}$ are all the same category, and $L, R : \mathcal{M} \to [\mathcal{M}, \mathcal{M}]$ are both the "left regular action" $A \cdot B = A \otimes B$. The *simplest* example is when this is specialized to the category **Set** with the cartesian monoidal product. In the general case, we think of the objects of $\mathcal{M}$ as encoding various kinds of "contexts" that we can extend objects of $\mathcal{C}$ or $\mathcal{D}$ with, and of the actions as the means of performing this extension. In this example, a "context" is just a set, and to extend a given set $A$ using a context $M$ is to take the product $M \times A$, so that elements of the extended set now carry extra data. In the general case, the fact that $\mathcal{M}$ is required to be monoidal and $L, R$ are required to be strong monoidal functors means that extending by a context $N$ and then by another context $M$ is the

same as extending by some "nested context" $M \otimes N$—$M \cdot N \cdot A \cong (M \otimes N) \cdot A$—so that we can see any series of extensions as a single large one. The fact that the action is by functors means that we can lift morphisms through extension-by-contexts: $A \to B$ to $M \cdot A \to M \cdot B$. In a concrete category, where morphisms are special functions, this means something like (in programming terms) "we can map over the original type within the context".

Given the above ingredients, we define a category $\mathbf{Optic}_L^R$. Its objects are pairs $(S \in \mathcal{C}, T \in \mathcal{D})$. The hom-sets are defined by a coend:

$$\mathbf{Optic}_L^R((S,T),(A,B)) = \int^{M \in \mathcal{M}} \mathcal{C}(S, M \cdot A) \times \mathcal{D}(M \cdot B, T). \tag{1}$$

Remembering that this is a particular quotient of the disjoint union

$$\bigsqcup_{M \in \mathcal{M}} \mathcal{C}(S, M \cdot A) \times \mathcal{D}(M \cdot B, T),$$

an optic $(S,T) \to (A,B)$ is an equivalence class of pairs of a $\mathcal{C}$-morphism $\mathsf{open} : S \to M \cdot A$ and a $\mathcal{D}$-morphism $\mathsf{close} : M \cdot B \to T$, with each pair allowed to use any $M \in \mathcal{M}$. Under the interpretation of objects of $M$ as "contexts", any representative of this equivalence class is a means of first dissecting $S$ into $A$ extended by some kind of context, hence possibly allowing some kind of transformation from $A$ to $B$ within the context, and then subsequently reconstituting the $M \cdot B$ into a final $T$. The equivalence imposed by the coend then enforces that all usage of optics must be "parametric" (more accurately, extranatural) in the particular $M$ provided, by identifying together $(\mathsf{open}, \mathsf{close})$ pairs that differ only by "what context they provide", rather than by "what the provided context is extending".

wait, is that right?

In the above-mentioned case of $\mathcal{C} = \mathcal{D} = \mathcal{M} = \mathbf{Set}$ with $L(M) = R(M) = M \times -$, this gives us *lenses*. If we instead equip $\mathcal{M} = \mathbf{Set}$ with $+$ as its monoidal product and use $L(M) = R(M) = M + -$, we get *prisms*.

write down the coend quotient explicitly?

Discuss composition of optics!

In the applications we discuss, we will be mostly interested in prosets. If we view the poset $\Omega$ as a thin category, and equip it with $\wedge$ as a monoidal product, we can identify prosets with $\Omega$-enriched categories. This turns out to give the right specialized-to-prosets version of notions such as presheaves, coends, and profunctors—and the proset version of these notions may be nontrivially different! This tends to arise in colimit-like situations, because colimits in $\Omega$ work differently from in **Set**—e.g., $\top \vee \top = \top$, while $1 + 1 \ncong 1$.[1] As an example of the impact this has: Colimits of presheaves are computed pointwise, so if $X$ is a non-empty proset, then $1 + 1 \ncong 1$ in $\mathrm{PSh}(X)$ (the category of ordinary presheaves), but $1 + 1 \cong 1$ in $\mathrm{PSh}_\Omega(X)$ (the category of $\Omega$-enriched presheaves).

In this setting, if we have prosets $\mathcal{C}, \mathcal{D}$, a monoidal proset $\mathcal{M}$, and strong monoidal monotone functions $L : \mathcal{M} \to [\mathcal{C}, \mathcal{C}], R : \mathcal{M} \to [\mathcal{D}, \mathcal{D}]$, then we can form an optic proset.[2] Working $\Omega$-enriched, the definition (1) reduces to

Should still work out a concrete example!

[1] In more detail, the subsingletons are not a coreflective subcategory of **Set**.

[2] Note that this is *not* in general the same as the optic category that results from interpreting

$$(S,T) \vdash_{\mathbf{Optic}_L^R} (A,B) \iff \exists M \in \mathcal{M}.\ S \vdash_{\mathcal{C}} M \cdot A \wedge M \cdot B \vdash_{\mathcal{D}} T. \qquad (2)$$

Thus, when $\Omega$-enriched it becomes slightly less meaningful to talk about "an optic"—rather, the question becomes whether a particular inequality/entailment holds in the optic proset.

# 3 Basic Optics for Separation Logic: Ramification

## 3.1 Optics for the Left Regular Action

The description of what (the above formulation of) an optic allows—decomposition into a subpart extended by an abstracted context, manipulation of the subpart, and then reconstitution to a modified whole—applies well as a description of how a number of separation logic reasoning techniques operate on knowledge of program state. Indeed, some of these techniques can be shown to fit within the framework of optics!

How strong of a claim do I want to make here?

Fix a separation logic that admits typical intuitionistic propositional logic and includes $-\!\!*$. The assertions of the logic form a proset under $\vdash$; call this proset $\mathcal{S}$. $\mathcal{S}$ is bicartesian closed, and it is symmetric closed monoidal under $(*, \mathsf{emp}, -\!\!*)$.

Wait, emp being a unit is *not* a universal assumption!

We start by considering the simplest kind of optics involving $\mathcal{S}$. First, for *any* monoidal proset $\mathcal{M}$, define the *left regular action* $\mathrm{LReg}_{\mathcal{M}} : \mathcal{M} \to [\mathcal{M}, \mathcal{M}]$ of $\mathcal{M}$ on itself to be (as previously discussed) $\mathrm{LReg}_{\mathcal{M}}(A) = A \otimes -$. We can then form an optic proset with $\mathcal{C} = \mathcal{D} = \mathcal{M} = \mathcal{S}$ and $L = R = \mathrm{LReg}_{\mathcal{S}}$. We will call this proset $\mathbf{Ram}_{\mathcal{S}}$, because it gives rise to *ramification* as in "The Ramifications of Sharing in Data Structures". By the simplified coend formula (2), we have

cite

$$(S,T) \vdash_{\mathbf{Ram}_{\mathcal{S}}} (A,B) \iff \exists M \in \mathcal{S}.\ S \vdash M * A \wedge M * B \vdash T.$$

We can correspond this to a particular separation logic entailment:

$$(S,T) \vdash_{\mathbf{Ram}_{\mathcal{S}}} (A,B) \iff S \vdash (B -\!\!* T) * A, \qquad (3)$$

which is recognizable as the premise of the RAMIFY rule. This equivalence is not hard to prove directly, but it is also an instance of a more general phenomenon whereby optics can have "concrete representations" as morphisms in $\mathcal{C}$. In particular, if we have some $\mathcal{C}, \mathcal{D}, \mathcal{M}, L, R$, some $(S,T), (A,B) \in \mathbf{Optic}_L^R$, and the functor $- \cdot B : \mathcal{M} \to \mathcal{D}$ admits a right adjoint $R_B : \mathcal{D} \to \mathcal{M}$, then there is a chain of isomorphisms

cite riley—this is lifted directly from p31 w/ just small tweaks!

the ingredients above as ordinary categories, because that optic category need not be thin—this optic proset can be acquired as the quotient of that category which identifies together all parallel morphisms.

$$\begin{aligned}\mathbf{Optic}_L^R((S,T),(A,B)) &= \int^{M\in\mathcal{M}} \mathcal{C}(S, M\cdot A)\times\mathcal{D}(M\cdot B, T) \text{ (definition)}\\ &\cong \int^{M\in\mathcal{M}} \mathcal{C}(S, M\cdot A)\times\mathcal{M}(M, R_B(T)) \text{ (adjunction)}\\ &\cong \mathcal{C}(S, R_B(T)\cdot A) \text{ (Ninja Yoneda lemma).} \qquad (*)\end{aligned}$$

In the $\mathbf{Ram}_{\mathcal{S}}$ case, we have $(-\cdot B) = (-*B) \dashv (B \multimap -)$. Setting $R_B = (B \multimap -)$ in (*) and otherwise instantiating it with the machinery associated to $\mathbf{Ram}_{\mathcal{S}}$, we recover (3). This is perhaps fancier than necessary for a single simple case, but recognizing its applicability to many kinds of optics will subsequently be useful.

## 3.2 Tambara Modules and Applying Optics to Hoare Triples

The derivation of the RAMIFY rule from the FRAME rule, too, can be seen as a case of a more general phenomenon. First, we will need yet another representation of optics, in terms of *Tambara modules*.

This representation can be motivated by switching perspectives on what having a given optic fundamentally enables us to do, from a step-by-step viewpoint to a big-picture one. If we have some "transformation from $A$ to $B$" or "relationship between $A$ and $B$" which can be be used even when $A$ and $B$ are extended by some additional context $M \in \mathcal{M}$, then an optic $(S,T) \to (A,B)$ allows us to lift to a "transformation" or "relationship" of the same kind between $S$ and $T$.[3] As a fairly literal example, whenever $\mathcal{C} = \mathcal{D}$ and $L = R$—such as in the case of lenses or $\mathbf{Ram}_{\mathcal{S}}$—ordinary morphisms are just such a kind of "transformation": whenever $f : A \to B$, we can extend to $L(M)(f) : M \cdot A \to M \cdot B$.[4] More generally, Tambara modules are the appropriate formalism for "some notion of transformation or relationship which can be used even within contexts", and so optics turn out to correspond to natural "lifting operations" that operate on any Tambara module.

This footnote is slightly bullshit—need to note that the 2nd argument has flipped variance.

Any sources on *mixed* Tambara modules, or am I gonna have to double check for myself that the arguments still go through?

Formally: Recall that a *profunctor* $P : \mathcal{D} \nrightarrow \mathcal{C}$ is a functor $P : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$; or equivalently, by swapping the arguments and currying, a functor $P' : \mathcal{D} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] = \mathrm{PSh}(\mathcal{C})$.[5] A particular such profunctor fixes, for our purposes, a concept of "transformation" or "relation" from objects of $\mathcal{C}$ to objects of $\mathcal{D}$: for $A \in \mathcal{C}, B \in \mathcal{D}$, we can view $P(A,B)$ as a "generalized hom-set"—the set of "heteromorphisms" $A \rightsquigarrow B$. Then, the action of $P$ on morphisms is "composition of morphisms with heteromorphisms"—although this interpretation only goes

[3] If the variance seems backward at first glance, note that the action of a morphism $X \to Y$ (in any category) on morphisms out of $Y$ is to "lift" them to morphisms out of $X$.

[4] Note that $L = R$ is necessary here, since we must put $S \to L(M)(A)$ and $R(M)(B) \to T$ on either side of the extended morphism.

[5] By the Yoneda lemma, another relationship between these two forms can be given by $P(A,B) \cong \mathrm{PSh}(\mathcal{D})(y(A), P'(B))$ (where $y$ is the Yoneda embedding), which provides some justification to viewing a profunctor as a kind of tweaked hom-functor.

so far, since a profunctor does not provide a way of composing its heteromorphisms with *each other*, even in cases where the types happen to line up. Next, given a monoidal category $\mathcal{M}$ with actions $L$ on $\mathcal{C}$ and $R$ on $\mathcal{D}$, we would like to consider those kinds of heteromorphism that can be lifted through the action of $\mathcal{M}$, since these are then the ones which can be lifted using an optic. Given $P$ as above, $P$ is turned into a *Tambara module* $(\mathcal{D}, R) \nrightarrow (\mathcal{C}, L)$ by equipping it with a family of functions that perform this "lifting",

Cite!

$$\zeta_{A,B,M} : P(A, B) \to P(M \cdot A, M \cdot B),$$

subject to several conditions regarding naturality, dinaturality, and compatibility with the action. However, these conditions are trivial in the $\Omega$-enriched version we will primarily work with, so we will not cover them in detail here.

At this point, we can now examine the precise connection with optics. Suppose $P$ is indeed equipped as a Tambara module, and we have a heteromorphism $h : A \rightsquigarrow B$. Suppose further we have an optic $o \in \mathbf{Optic}_L^R((S, T), (A, B))$ with some representative $(\mathsf{open} : S \to M \cdot A, \mathsf{close} : M \cdot B \to T)$. We can first take $\zeta_{A,B,M}(h) : M \cdot A \rightsquigarrow M \cdot B$, and then use profunctoriality to put $\mathsf{open}$ and $\mathsf{close}$ before and after, giving

$$P(\mathsf{open}, \mathsf{close})(\zeta_{A,B,M}(h)) : S \rightsquigarrow T.$$

This turns out to be independent of the choice of representative of $o$, and so it is well-defined as an action of $\mathbf{Optic}_L^R((S, T), (A, B))$ on $P(A, B)$. Then, vitally, there is a Yoneda-esque result that this mapping from $A \rightsquigarrow B$ to $S \rightsquigarrow T$ for all Tambara modules is *all* that characterizes an optic of this type! In particular: There is a notion of "morphism of Tambara modules" such that the Tambara modules $(\mathcal{D}, R) \nrightarrow (\mathcal{C}, L)$ are the objects of a category, and then for $A \in \mathcal{C}, B \in \mathcal{D}$ we have a functor $\mathrm{ev}_{A,B}$ from this category of Tambara modules to **Set** which evaluates its argument at $(A, B)$—i.e., it takes the set of heteromorphisms $A \rightsquigarrow B$. Then, natural transformations from $\mathrm{ev}_{A,B}$ to $\mathrm{ev}_{S,T}$ correspond exactly to optics $(S, T) \to (A, B)$.

Pull out into a theorem statement, and cite!

We now turn to the $\Omega$-enriched analogues of these definitions. An $\Omega$-enriched profunctor $P : \mathcal{D} \nrightarrow \mathcal{C}$ is a monotone function $P : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \Omega$, which we can view as "generalized entailment". A vital example for us will be that, for any command $C$, we have $H_C : \mathcal{S} \nrightarrow \mathcal{S}$ defined by $H_C(P, Q) = \{P\}\, C\, \{Q\}$—the classic Hoare logic rule of consequence precisely states that $H_C$ is an $\Omega$-enriched profunctor! Moving forward, we will tend to drop the "$\Omega$-enriched" prefix from "$\Omega$-enriched profunctor" when it is established that $\mathcal{C}$ and $\mathcal{D}$ are prosets, since we have little need for the other kind in that case.

Probably worth revising somewhere to say sth like "$\Omega$-enriched definitions are usually just special cases of the standard definition".

In the $\Omega$-enriched case, the equipment of a Tambara module reduces to a property that a profunctor must satisfy:

$$\mathbf{Tamb}(P) \iff \forall A \in \mathcal{C},\, B \in \mathcal{D},\, M \in \mathcal{M}.\; P(A, B) \to P(M \cdot A, M \cdot B), \quad (4)$$

where $\to$ is just implication. Similarly, the Tambara module-based formulation of the preorder on the optic proset reduces to

Cram in somewhere a mention of the nature of Tambara module homomorphisms or

$$(S,T) \vdash_{\mathbf{Optic}_L^R} (A,B) \iff \forall P.\ \mathbf{Tamb}(P) \to P(A,B) \to P(S,T). \quad (5)$$

We can now put this to work in a very simple case. If we instantiate (4) with $\mathcal{S}$ under the left regular action, we get

$$\mathbf{Tamb}(P) \iff \forall A, B, M \in \mathcal{S}.\ P(A,B) \to P(M * A, M * B).$$

Now consider $\mathbf{Tamb}(H_C)$:

$$\mathbf{Tamb}(H_C) \iff \forall A, B, M \in \mathcal{S}.\ \{A\}\, C\, \{B\} \to \{M * A\}\, C\, \{M * B\}.$$

In traditional separation logic, we have $\mathbf{Tamb}(H_C)$ only for $C$ which does not modify any variables, but in systems such as Iris, or those that use variables-as-resources, the content of the frame rule is exactly that $H_C$ is a Tambara module $(\mathcal{S}, \mathrm{LReg}_{\mathcal{S}}) \nrightarrow (\mathcal{S}, \mathrm{LReg}_{\mathcal{S}})$.[6]

Double check

Putting together the concrete representation formula with the Tambara module formulation, we can recover the RAMIFY rule based on the fact that $\mathbf{Tamb}(H_C)$:

$$\begin{aligned} S \vdash (B \nrightarrow T) * A &\iff (S,T) \vdash_{\mathbf{Ram}_{\mathcal{S}}} (A,B) \\ &\iff \forall P.\ \mathbf{Tamb}(P) \to P(A,B) \to P(S,T) \\ &\implies H_C(A,B) \to H_C(S,T) \\ &= \{A\}\, C\, \{B\} \to \{S\}\, C\, \{T\}. \end{aligned}$$

Once again, this is considerably more machinery than is necessary to derive such a rule—what is important is that similar reasoning can allow the uniform derivation of a broad range of similar-in-spirit rules based on the categorical structure of the ingredients.

# 4 Building New Kinds of Optic: Quantified Ramification

The Verified Software Toolchain includes a file called `ramification_lemmas.v`. One of the modules in this file, `RAMIF_Q`, deals with a version of ramification where the premise takes the form $S \vdash (\forall x : X.\ B(x) \nrightarrow T(x)) * A$, for some type $X$. In this section, we gather some constructions and facts (many fairly standard) which allow this class of formula—and others like it—to be easily derived as the concrete representation of a flavor of optic.

citation needed

Also Tambara module stuff coming up.

- Given any set $X$, we can form the *discrete proset* $\delta X$ whose preorder is just equality. If $\mathcal{C}$ is any proset, then monotone functions $\delta X \to \mathcal{C}$ are the same as ordinary functions $X \to \mathcal{C}$.[7]

[6] We will later recover the traditional frame rule as the statement that $H_C$ is a Tambara module for a different pair of actions.

[7] That is, $\delta$ is left adjoint to the forgetful functor from prosets to sets.

- If $\mathcal{C}, \mathcal{D}$ are prosets and $\mathcal{D}$ is monoidal, then $[\mathcal{C}, \mathcal{D}]$ can be made monoidal under the pointwise product, with unit the constant function at $\mathcal{D}$'s unit. If $\mathcal{D}$ is symmetric monoidal, then so is the pointwise product. If $\mathcal{D}$ is closed monoidal, then the internal hom does not necessarily lift pointwise, because the variances are wrong, but if $\mathcal{C}$'s relation is symmetric, then it does (since then the variances can be flipped). This is especially important in the case where $\mathcal{C} = \delta X$. When $\mathcal{D} = \mathcal{S}$, we will write these pointwise operations with the same $*$ and $\multimap$.

Double check?

- Suppose a proset $\mathcal{C}$ has all limits (resp. colimits) of shape $\mathcal{J}$. Then $\lim_{\mathcal{J}} : [\mathcal{J}, \mathcal{C}] \to \mathcal{C}$ (resp. $\mathrm{colim}_{\mathcal{J}}$) is monotone. Define $K_{\mathcal{J}} : \mathcal{C} \to [\mathcal{J}, \mathcal{C}]$ by $K_{\mathcal{J}}(A)(J) = A$; this is easily also monotone, and we have $K_{\mathcal{J}} \dashv \lim_{\mathcal{J}}$ (resp. $\mathrm{colim}_{\mathcal{J}} \dashv K_{\mathcal{J}}$). In the case where $\mathcal{J} = \delta X$, these are $X$-indexed infima and suprema, and we write $\lim_{\delta X}$ as $\forall_X$ and $\mathrm{colim}_{\delta X}$ as $\exists_X$, as well as abbreviating $K_{\delta X}$ to just $K_X$.

- If $\mathcal{C}$ is monoidal, then $K_{\mathcal{J}}$ is strong monoidal. If $\lim_{\mathcal{J}} : [\mathcal{J}, \mathcal{C}] \to \mathcal{C}$ exists, it is lax monoidal. If $\mathrm{colim}_{\mathcal{J}} : [\mathcal{J}, \mathcal{C}] \to \mathcal{C}$ exists, it is oplax monoidal.

Next paragraph probably needs a lot of work on the exposition.

Suppose $\mathcal{S}$ has all $X$-indexed infima, so that we have a sensible interpretation of the formula $\forall x : X.\ \varphi(x)$ as $\forall_X(x \mapsto \varphi(x))$. We now consider what kind of optic would give rise to quantified ramification. First of all, $S$ and $A$ are still assertions, but $B$ and $T$ are now assertion-valued *predicates* parameterized over some $X$. Thus, we still have $\mathcal{C} = \mathcal{S}$, but for $B, T$ to be objects of $\mathcal{D}$, we must switch to $\mathcal{D} = [\delta X, \mathcal{S}]$. Next, we must find an $\mathcal{M}$ and actions $L : \mathcal{M} \to [\mathcal{S}, \mathcal{S}], R : \mathcal{M} \to [[\delta X, \mathcal{S}], [\delta X, \mathcal{S}]]$ so that we get the desired concrete representation. Recalling (*), we need $R(-)(B) : \mathcal{M} \to [\delta X, \mathcal{S}]$ to have a right adjoint $R_B : [\delta X, \mathcal{S}] \to \mathcal{M}$ for each $B$, and we need

$$S \vdash \forall_X(B \multimap T) * A \iff S \vdash R_B(T) \cdot A.$$

This is easily satisfied if $\mathcal{M} = \mathcal{S}$, $L = \mathrm{LReg}_{\mathcal{S}}$, and $R_B = \forall_X \circ (B \multimap -)$. Then $K_X \dashv \forall_X$ and $(- * B) \dashv (B \multimap -)$, so $(- * B) \circ K_X \dashv R_B$, and we must have $R(-)(B) = (- * B) \circ K_X$. Finally, we can rewrite this to $R(M) = B \mapsto K_X(M) * B = \mathrm{LReg}_{[\delta X, \mathcal{S}]}(K_X(M))$ and then $R = \mathrm{LReg}_{[\delta X, \mathcal{S}]} \circ K_X$, which is a composition of strong monoidal monotone functions and hence itself a strong monoidal monotone function. We define $\mathbf{RamQ}_{\mathcal{S},X}$ to be the optic proset for the actions $\mathrm{LReg}_{\mathcal{S}}$ and $\mathrm{LReg}_{[\delta X, \mathcal{S}]} \circ K_X$, so by the reasoning just given, we have

$$(S, T) \vdash_{\mathbf{RamQ}_{\mathcal{S},X}} (A, B) \iff S \vdash (\forall x : X.\ B(x) \multimap T(x)) * A.$$

We next gather some results on profunctors and Tambara modules for similar purposes.

- Composition of profunctors $P : \mathcal{E} \nrightarrow \mathcal{D}$ and $Q : \mathcal{D} \nrightarrow \mathcal{C}$ is defined by the coend

on objects...

$$(Q \circ P)(C, E) = \int^{D \in \mathcal{D}} Q(C, D) \times P(D, E).$$

For Ω-enriched profunctors, this reduces to ordinary composition of relations:

$$(Q \circ P)(C, E) = \exists D \in \mathcal{D}.\ Q(C, D) \wedge P(D, E).$$

This is associative, and has identity $\mathrm{id}_{\mathcal{C}}(C, C') = C \vdash C'$, so we have a category with prosets as objects and profunctors as morphisms.

- If $F : \mathcal{D} \to \mathcal{C}$ is a monotone function, then we can define a profunctor $F_* : \mathcal{D} \nrightarrow \mathcal{C}$ by $F_*(A, B) = A \vdash F(B)$ and a profunctor $F^* : \mathcal{C} \nrightarrow \mathcal{D}$ by $F^*(B, A) = F(B) \vdash A$. For $G : \mathcal{C} \to \mathcal{D}$, we have $F \dashv G$ iff $F^* \cong G_*$. If $P : \mathcal{C} \nrightarrow \mathcal{E}$, then $P \circ F_* \cong (E, D) \mapsto P(E, F(D))$, and if $Q : \mathcal{E} \nrightarrow \mathcal{C}$, then $F^* \circ Q \cong (D, E) \mapsto Q(F(D), E)$.

- Suppose we have a monoidal proset $\mathcal{M}$ and actions $L : \mathcal{M} \to [\mathcal{C}, \mathcal{C}], I : \mathcal{M} \to [\mathcal{D}, \mathcal{D}], R : \mathcal{M} \to [\mathcal{E}, \mathcal{E}]$. If $P$ is a Tambara module $(\mathcal{E}, R) \nrightarrow (\mathcal{D}, I)$ and $Q$ is a Tambara module $(\mathcal{D}, I) \nrightarrow (\mathcal{C}, L)$, then $Q \circ P$ is a Tambara module $(\mathcal{E}, R) \nrightarrow (\mathcal{C}, L)$. Additionally, as informally alluded to, the identity profunctor on any $\mathcal{C}$ is always a Tambara module $(\mathcal{C}, F) \nrightarrow (\mathcal{C}, F)$ for any action $F$ on $\mathcal{C}$. Altogether, then, for any fixed monoidal proset $\mathcal{M}$, we can form a category whose objects are prosets equipped with an action of $\mathcal{M}$, and whose morphisms are Tambara modules.[8] This has a faithful forgetful functor to the aforementioned profunctor category.

- Suppose $F : \mathcal{D} \to \mathcal{C}$ and $L : \mathcal{M} \to [\mathcal{C}, \mathcal{C}], R : \mathcal{M} \to [\mathcal{D}, \mathcal{D}]$. Then $F_* : \mathcal{D} \nrightarrow \mathcal{C}$ is a Tambara module $(\mathcal{D}, R) \nrightarrow (\mathcal{C}, L)$ iff $F$ is "lax-equivariant":

  $$\forall M \in \mathcal{M}, B \in \mathcal{D}. \quad M \cdot F(B) \vdash F(M \cdot B),$$

  and $F^* : \mathcal{C} \nrightarrow \mathcal{D}$ is a Tambara module $(\mathcal{C}, L) \nrightarrow (\mathcal{D}, R)$ iff $F$ is "oplax-equivariant":

  $$\forall M \in \mathcal{M}, B \in \mathcal{D}. \quad F(M \cdot B) \vdash M \cdot F(B).$$

Next bit is motivated reasoning and total bullshitting hmm. Also, need to not fucking say "obvious"!!!!!!

With these results in hand, we can work out the appropriate Hoare logic rule[s] for eliminating an entailment in $\mathbf{RamQ}_{\mathcal{S},X}$, and hence a quantified ramification—we just need to find a Hoare-triple-based based Tambara module $([\delta X, \mathcal{S}], \mathrm{LReg}_{[\delta X, \mathcal{S}]} \circ K_X) \nrightarrow (\mathcal{S}, \mathrm{LReg}_{\mathcal{S}})$. We start with $H_C : (\mathcal{S}, \mathrm{LReg}_{\mathcal{S}}) \nrightarrow (\mathcal{S}, \mathrm{LReg}_{\mathcal{S}})$, so the most obvious thing to do is compose it with some other Tambara module already of the desired type. The easiest way to get a profunctor

[8] Actually, we can form a category with these same objects but whose morphisms are "lax-equivariant" *monotone functions* between those prosets instead of profunctors; then PSh gives rise to a monad on this category, and the Tambara module category arises as the Kleisli category for this monad.

Currently bullshitting!! Double check this!!

$[\delta X, \mathcal{S}] \nrightarrow \mathcal{S}$ is by taking $F^*$ or $F_*$ of a monotone function; simultaneously, it is natural to expect a rule eliminating a quantified ramification to involve some kind of "Hoare triple with quantification", so the the most obvious candidate profunctors are $\forall_{X*}$ and $\exists_{X*}$. To check that these are Tambara modules, we just need to check that $\forall_X$ and $\exists_X$ are "lax-equivariant" as above, which expands to the following two separation logic validities:[9]

$$M * \forall_X(B) \vdash \forall_X(x \mapsto M * B(x))$$
$$M * \exists_X(B) \vdash \exists_X(x \mapsto M * B(x)).$$

Should I cut out this diversion?

We conclude that

$$H_C \circ \forall_{X*},\ H_C \circ \exists_{X*} : ([\delta X, \mathcal{S}], \mathrm{LReg}_{[\delta X, \mathcal{S}]} \circ K_X) \nrightarrow (\mathcal{S}, \mathrm{LReg}_{\mathcal{S}}),$$

*in separation logics whose frame rule does not have side-conditions*

and note

$$H_C \circ \forall_{X*} \cong (P, Q) \mapsto \{P\}\, C\, \{\forall x.\ Q(x)\}$$
$$H_C \circ \exists_{X*} \cong (P, Q) \mapsto \{P\}\, C\, \{\exists x.\ Q(x)\}.$$

Altogether, we have

use sth better than \ frac

$$\frac{(S,T) \vdash_{\mathbf{RamQ}_{\mathcal{S},X}} (A,B) \qquad \{A\}\, C\, \{\forall x.B(x)\}}{\{S\}\, C\, \{\forall x.T(x)\}}$$

$$\frac{(S,T) \vdash_{\mathbf{RamQ}_{\mathcal{S},X}} (A,B) \qquad \{A\}\, C\, \{\exists x.B(x)\}}{\{S\}\, C\, \{\exists x.T(x)\}}$$

If we replace the $(S,T) \vdash_{\mathbf{RamQ}_{\mathcal{S},X}} (A,B)$ in the latter of the two rules above by expanding the concrete optic formula (*) to only the second-to-last step, we get the LOCALIZE rule from "Certifying Graph-Manipulating C Programs via Localizations within Data Structures".

This is not quite true—one of their points was management of the side condition. To be fair, this *does* work out like their point if you do the side-condition version of the frame rule in this framework, but that's not what I've said here.

um, it *is*, right?

## 5 Side Conditions for the Frame Rule

It is time to stop ignoring the fact that separation logics tend to have side conditions on their frame rule.

Given a command $C$, let $\mathcal{S} \setminus C$ denote the full subproset of $\mathcal{S}$ on assertions closed with respect to $C$'s modified variables—i.e., the subproset of assertions which satisfy the side condition of the frame rule for $C$—and let $\iota_C : \mathcal{S} \setminus C \hookrightarrow \mathcal{S}$ be the inclusion. We will need a basic assumption about our separation logic which is fairly universal: $\mathcal{S} \setminus C$ includes $\mathsf{emp}$ and is closed under $*$ and $-\!\!*$, so

[9] $\forall_{X*}$ remains a Tambara module if $\mathcal{S}$ is replaced with *any* monoidal proset with all $X$-indexed infima: by $K_X \dashv \forall_X$ we have $K_X^* \cong \forall_{X*}$, so $\forall_{X*}$ is a Tambara module iff $K_X$ is oplax-equivariant:

$$K_X(M * A) = K_X(M \cdot A) \vdash M \cdot K_X(A) = K_X(M) * K_X(A),$$

but this just asks that $K_X$ is oplax monoidal, which it is—indeed, it is *strong* monoidal! By contrast, the fact that $\exists_{X*}$ is a Tambara module hinges on the fact that $*$ distributes over $X$-indexed suprema, which follows from the existence of $-\!\!*$.

it inherits $\mathcal{S}$'s closed monoidal structure. Then $\iota_C$ is trivially strong monoidal. Therefore, we can restrict the left regular action to take contexts only from $\mathcal{S} \setminus C$ rather than from all of $\mathcal{S}$ using $\mathrm{LReg}_{\mathcal{S}} \circ \iota_C$. Then, the version of the frame rule with a side condition states that $H_C$ is a Tambara module $(\mathcal{S}, \mathrm{LReg}_{\mathcal{S}} \circ \iota_C) \nrightarrow (\mathcal{S}, \mathrm{LReg}_{\mathcal{S}} \circ \iota_C)$, and so optics for this action can be applied directly to Hoare triples for $C$ even when the frame rule has a side condition. We define $\mathbf{Ram}_{\mathcal{S} \setminus C}$ to be the optic proset for $\mathcal{C} = \mathcal{D} = \mathcal{S},\ \mathcal{M} = \mathcal{S} \setminus C,\ L = R = \mathrm{LReg}_{\mathcal{S}} \circ \iota_C$.

We can now recover the correct form of the LOCALIZE rule previously mentioned—we use the same ingredients for quantified ramification as before, except that we restrict the domain of the actions, so

$$\mathcal{C} = \mathcal{S},\ \mathcal{D} = [\delta X, \mathcal{S}],\ \mathcal{M} = \mathcal{S} \setminus C,\ L = \mathrm{LReg}_{\mathcal{S}} \circ \iota_C,\ R = \mathrm{LReg}_{[\delta X, \mathcal{S}]} \circ K_X \circ \iota_C.$$

This gives an optic proset which we will call $\mathbf{RamQ}_{\mathcal{S},X} \setminus C$.

Argue for $H_C \circ \exists_{X*}$ or something along those lines being a Tambara module.

Then the remark about partially expanding the concrete optic formula really does yield LOCALIZE, side condition and all.

no wait that's slightly off...

In order to work further with this subproset, we need one more fact about it whose proof rests on the details of the logic in question, but which is again true in most cases: $\mathcal{S} \setminus C$ is both reflective and coreflective in $\mathcal{S}$. That is, there are monotone functions $\lambda_C, \rho_C : \mathcal{S} \to \mathcal{S} \setminus C$ such that $\lambda_C \dashv \iota_C \dashv \rho_C$. Roughly speaking, $\lambda_C(P)$ is the existential closure of $P$ with respect to the modified variables of $C$, while $\rho_C(P)$ is the universal closure with respect to those variables.

Write up the proofs for an example case or two!

By some standard abstract nonsense, we can draw the following conclusions:

- $\lambda_C \circ \iota_C \cong \rho_C \circ \iota_C \cong \mathrm{id}_{\mathcal{S} \setminus C}$. I.e., for all $P \in \mathcal{S}_C$, $\lambda_C(\iota_C(P)) \dashv\vdash \rho_C(\iota_C(P)) \dashv\vdash P$.
- Define $\Diamond_C, \Box_C : \mathcal{S} \to \mathcal{S}$ by $\Diamond_C = \iota_C \circ \lambda_C$ and $\Box_C = \iota_C \circ \rho_C$ . Then $\Diamond_C$ is a monad, $\Box_C$ is a comonad, and $\Diamond_C \dashv \Box_C$.
- $\lambda_C$, $\iota_C$, and $\Diamond_C$ distribute over colimits (including $\vee$ and $\exists_X$) because they are left adjoints. $\rho_C$, $\iota_C$, and $\Box_C$ distribute over limits (including $\wedge$ and $\forall_X$) because they are right adjoints.
- An assertion $P$ is in $\mathcal{S} \setminus C$ iff it is an algebra of $\Diamond_C$ (i.e., $\Diamond_C P \vdash P$), or equivalently by the adjunction, a coalgebra of $\Box_C$ (i.e., $P \vdash \Box_C P$). Note that since $P \vdash \Diamond_C P$ and $\Box_C P \vdash P$ in general anyway, "algebra" and "coalgebra" coincide with "fixed point up to logical equivalence".[10]
- $\lambda_C$ and $\Diamond_C$ are oplax monoidal, and $\rho_C$ and $\Box_C$ are lax monoidal.

Check whether any are actually strong monoidal, hm.

[10] Of course, if we were working in arbitrary categories instead of prosets, algebras of an arbitrary monad (resp. coalgebras of an arbitrary comonad) would not need to be inverse to the monad's unit (resp. comonad's counit). But coincidentally, in the case we are considering of a reflective (resp. coreflective) subcategory, they *would* be inverse!

This lets us derive the concrete representation for $\mathbf{Ram}_{\mathcal{S}} \backslash C$ and $\mathbf{RamQ}_{\mathcal{S},X} \backslash C$. For the former, if $B \in \mathcal{S}$,

$$(- \cdot B) = (- * B) \circ \iota_C \dashv \rho_C \circ (B \mathrel{-\!\!*} -).$$

Plugging this into (*) gives

$$\begin{aligned}(S,T) \vdash_{\mathbf{Ram}_{\mathcal{S}} \backslash C} (A,B) &\iff S \vdash \iota_C(\rho_C(B \mathrel{-\!\!*} T)) * A \\ &= S \vdash \Box_C(B \mathrel{-\!\!*} T) * A.\end{aligned}$$

If $B$ and $T$ already satisfy the side condition, then $\Box_C(B \mathrel{-\!\!*} T) \dashv\vdash B \mathrel{-\!\!*} T$, and in this case the concrete representation is the same as for $\mathbf{Ram}_{\mathcal{S}}$.

Next, take $\mathbf{RamQ}_{\mathcal{S}} \setminus C$. Suppose $B \in [\delta X, \mathcal{S}]$. Then

$$(- \cdot B) = (- * B) \circ K_X \circ \iota_C \dashv \rho_C \circ \forall_X \circ (B \mathrel{-\!\!*} -).$$

Once again, we apply (*):

$$\begin{aligned}(S,T) \vdash_{\mathbf{RamQ}_{\mathcal{S},X} \backslash C} (A,B) &\iff S \vdash \iota_C(\rho_C(\forall_X(B \mathrel{-\!\!*} T))) * A \\ &\iff S \vdash (\forall x : X.\ \Box_C(B(x) \mathrel{-\!\!*} T(x))) * A.\end{aligned}$$

As before, if $B(x)$ and $T(x)$ already satisfy the side condition for all $x$, then $\forall x : X.\ \Box_C(B(x) \mathrel{-\!\!*} T(x)) \dashv\vdash \forall x : X.\ B(x) \mathrel{-\!\!*} T(x)$, and in this case the concrete representation is the same as for $\mathbf{RamQ}_{\mathcal{S},X}$.