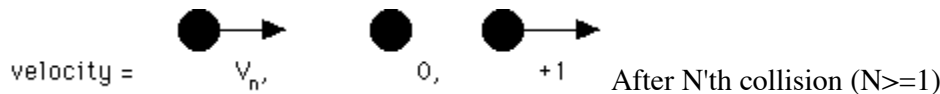


# One-Dimensional Billiards

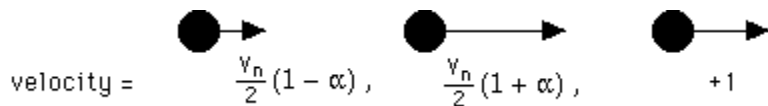
S. Glashow mentioned this problem during a talk I heard him give at the Nov `92 AAPT meeting at Vassar. He said that it's possible to have an infinite number of collisions if the "elasticity" (Glashow's term) is less than  $\sqrt{49} - \sqrt{48}$ . Here's how it works.

Suppose that at least one collision has already taken place, and that the last collision was between the two balls on the right, which must therefore be moving apart. Choose a velocity scale and inertial frame such that the velocity of the middle ball is 0 and the right ball is +1. Call the left velocity  $V_n$ , as shown in this picture.



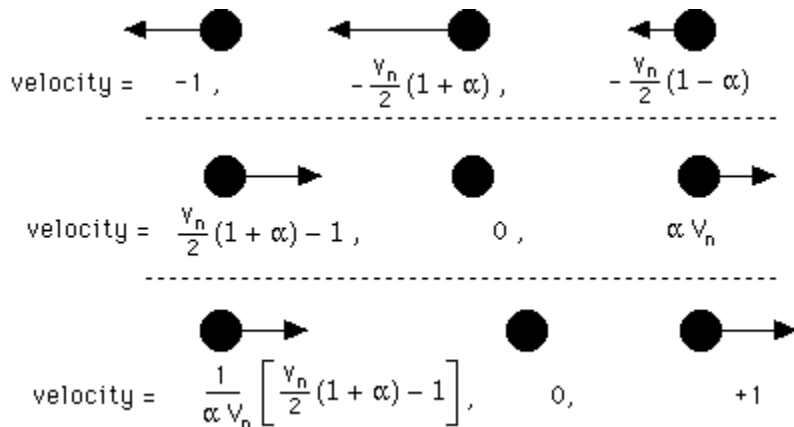
The next collision of the two balls on the left will happen only if  $V_n > 0$ .

Assuming that  $V_n$  is positive, the situation after collision ( $N+1$ ) can be found by momentum conservation. We get



where  $0 \leq \alpha \leq 1$  is the "elasticity."  $\alpha = 1$  is a perfectly elastic collision in which the ball on the left stops and the middle ball continues on at  $V_n$ , while  $\alpha = 0$  is a perfectly inelastic collision in which the two balls stick together and both move at  $\frac{V_n}{2}$ .

Now, to continue, 1) reverse left and right, 2) move into the rest frame of the middle ball, and 3) scale the velocities so that once again the ball on what will be the right is moving at velocity +1. These three steps are shown below.



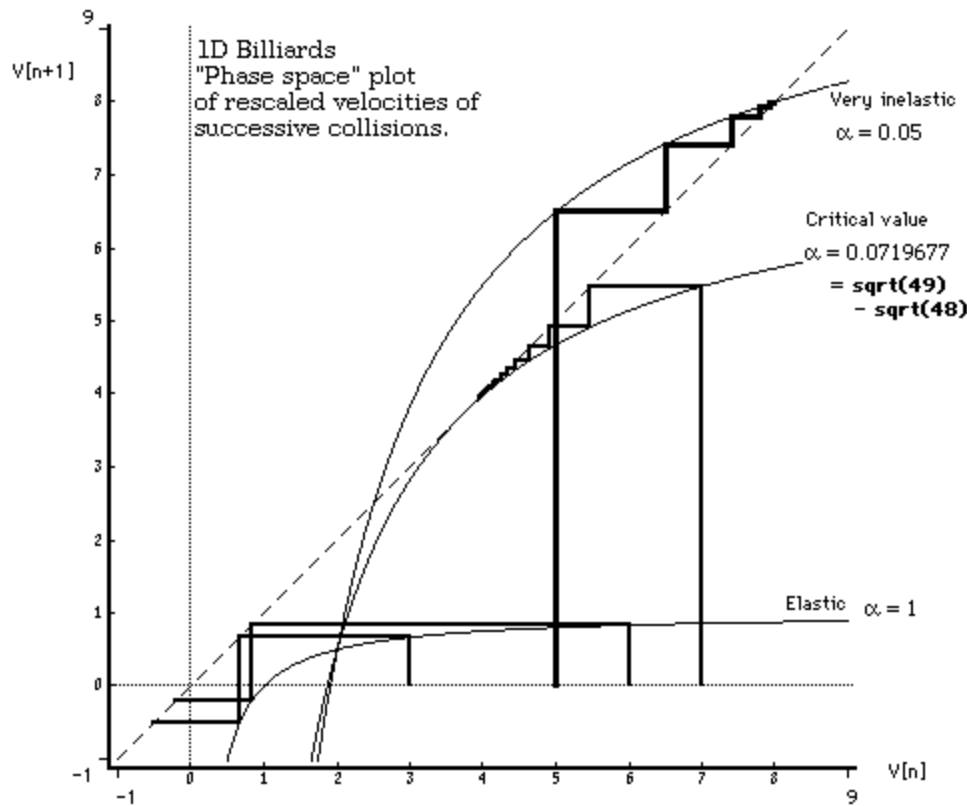
The problem now looks just like it did in the beginning, except that  $V_n$  (after the n'th collision) has changed to a new velocity  $V_{n+1}$  (the velocity after collision n+1), where

$$V_{n+1} = \frac{1}{2} + \frac{1}{\alpha} \left[ \frac{1}{2} - \frac{1}{V_n} \right]$$

As this relation is applied repeatedly for increasing  $n$ , collisions will continue until  $V_{n+1} < 0$ .

To visualize what will happen in the long run we can borrow a graphical technique from chaos studies. Plot  $V_{n+1}$  vs.  $V_n$ , as shown below. Starting at some value for  $V_n$ , we find the next by traveling vertically to the curve given above to find  $V_{n+1}$ . We then move horizontally to the  $V_{n+1} = V_n$  dotted line to find this new velocity on the x axis, and then repeat the process, zig-zagging between the  $V_{n+1}$  vs.  $V_n$  curve and the  $V_{n+1} = V_n$  line.

Curves are shown below for three different values of  $\alpha$ ; 1, 0.072, and 0.05. In each case the first collision (in which all three balls are approaching each other) does not fit our description and is not shown. For values of  $\alpha$  below the critical value shown where the two curves just touch, a stable solution exists and collisions can continue on forever.



To find the critical value where stable solutions begin, we find the values of  $\alpha$  where  $V_n = V_{n+1}$  is allowed.

Therefore, letting  $V = V_n = V_{n+1}$  in the equation at the top of the last page, we have this equation, which we want to solve, if possible.

$$V \stackrel{?}{=} \frac{1}{2} + \frac{1}{\alpha} \left[ \frac{1}{2} - \frac{1}{V} \right]$$

$$V \stackrel{?}{=} \frac{1+\beta}{2} - \frac{\beta}{V}, \quad \text{where } \beta = \frac{1}{\alpha}$$

As shown in the graph on the previous page, only some values of *alpha* (or *beta*) allow a solution.

Multiplying through by  $V$ , collecting terms, and completing the square on  $V$  gives

$$\left[ V - \frac{1+\beta}{4} \right]^2 \stackrel{?}{=} \beta - \left[ \frac{1+\beta}{4} \right]^2$$

which can only be true when

$$\beta - \left[ \frac{1+\beta}{4} \right]^2 \geq 0$$

$$\beta^2 - 14\beta + 1 \geq 0$$

$$\beta \leq \frac{14 - \sqrt{14^2 - 4}}{2} \quad \text{or} \quad \beta \geq \frac{14 + \sqrt{14^2 - 4}}{2}.$$

Since  $\alpha \leq 1$ , and  $\beta = 1/\alpha$ , we must have  $\beta \geq 1$ , so only the second range is possible. Thus possible values where  $V_n = V_{n+1}$  are for

$$\beta \geq 7 + \sqrt{48}$$

$$\alpha \leq \frac{1}{7 + \sqrt{48}} = 7 - \sqrt{48}$$

or

$$\alpha \leq \sqrt{49} - \sqrt{48}$$

At this critical *alpha*, the velocity of the ball on the left must be  $(1+\beta)/4 = 3.732$  times the velocity on the far right, in the rest frame of the middle ball.

So there you go.

By the way, the plot was made with the following routine in NumberCrunch II.

```
# This is V[n+1] as a function f of V[n] and Alpha.
NextV(v,alpha) = 0.5 + 1/alpha ( 0.5 - 1/v);
# The magic value where infinitely collisions are possible:
C = 7 - sqrt(48); # which is = 0.07179677
# This returns a 2 by N array of points defining the line
# segments that trace the path of N collisions in the V[n+1]
```

```

# vs V[n] phase plot.
FUNCTION IteratePath(V1, Alpha)
. VAR Done, j, Vn, Vnp1, Path
. BEGIN
. Done = 0
. Path[1... 2, 1... 31] = 0
. Path[1... 2, 1] = [V1, 0]
. j = 0
. Vn = V1<
. WHILE Done<>1 DO
. BEGIN
. j = j + 2
. Vnp1 = NextV( Vn, alpha)
. Path[1... 2, j] = [Vn, Vnp1] # Move to the NextV curve
. Path[1... 2, j+1] = [Vnp1, Vnp1] # Move to the x=y line.
. IF Vnp1 < 0 THEN
. Done = 1 # Stop if velocity has gone negative.
. ELSE IF j > 27 THEN
. Done = 1 # or we've done lots of velocities
. ELSE IF |Vnp1 - Vn| < 1e-4 THEN
. DONE = 1 # or we've hit a stable point.
. Vn = Vnp1 # Move on to the next.
. END

. IteratePath = Path[1... 2, 1... (j+1)]
. END
.

# For example, put
# IteratePath(5, 0.05)
# in the [x,y] field of a plot item to graph the thickest black line seen in the plot.

```

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