Probability

A probability experiment is a process that leads to well-defined results called outcomes.

An outcome is a result of a single trial of a probability experiment

e.g. of Probability experiments are the following processes :-

- flipping a coin,
- rolling a die,
- drawing a card from a deck, etc.

Examples of trials :-

- flipping a coin once,
- rolling one die once,
- drawing a single card from a deck, etc.

A sample space is the set of all possible outcomes of a probability experiment

Experiment	Sample space
Toss one coin	{head, tail}
Roll a die	{1, 2, 3, 4, 5, 6}
A game of Rock, Paper, Scissors between 2 people	{ Rock/Rock, Rock/Paper, Rock/Scissors, Paper/Rock, Paper/Paper, Paper/Scissors, Scissors/Rock, Scissors/Paper, Scissors/Scissors}
Toss two coins	{head/head, tail/tail, head/tail, tail/head}.

Note: Consider the game of Rock, Paper, Scissors between two people, called Player A and Player B say. It is clear that the outcome Player A, Rock, and Player B, Scissors, is different from the outcome Player A, Scissors, and Player B, Rock.

A similar argument can be made for the outcomes head/tail and tail/head being different in the toss two coin experiment if we consider one coin to be a quarter and the other coin to be a dime.

An <u>event</u> is any subset of the sample space of a probability experiment.

e.g.

- When a die is rolled, we may consider obtaining an even number, ie 2, 4, or 6. Getting an even number when rolling a die is an example of an event.

- When two coins are tossed:

  event : the coins show different sides

  ie {heads/tails, tails/heads}.

Classical probability

- assumes that all outcomes in the sample space are equally likely to occur.

<u>Formula for Classical Probability</u>

The probability of an event E is

$$\frac{\text{number of outcomes in } E}{\text{total number of outcomes in the sample space } S}$$

This probability is denoted by

$$P(E) = \frac{n(E)}{n(S)}$$

This probability, called classical probability, is based on a sample space S.

Question.

1. A die is rolled. Find the probability of getting:-

(a) A 2.

(b) A number less than 5

(c) An odd number.

2. Two coins are tossed. Find the probability of getting:-

(a) Two heads

(b) At least one head

(c) At most one head.

Solution

Sample space is 1, 2, 3, 4, 5 and 6. So $n(S) = 6$.

1. (a) There is only one outcome that gives a 2, so $P(2) = 1/6$

(b) There are four possible outcomes for the event of getting less than 5, namely 1, 2, 3 or 4.

So $n(E) = 4$

and $P(\text{a number less than } 5) = \dfrac{n(E)}{n(S)} = \dfrac{4}{6} = \dfrac{2}{3}$

(c) There are three possible outcomes for the event of getting an odd number,

namely 1, 3, or 5 : hence $n(E) = 3$

$P(\text{odd number}) = \dfrac{n(E)}{n(S)} = \dfrac{3}{6} = \dfrac{1}{2}$

2. The sample space is $\{HH, HT, TH, TT\}$. Hence $n(S) = 4$

(a)

There is only way to get two heads. ie HH.

Hence $P(\text{two heads}) = \dfrac{n(E)}{n(S)} = \dfrac{1}{4}$

(b) "At least one head" means one or more heads, ie one head or two heads.

There are three ways: HT, TH and HH. So $n(E) = 3$.

Hence

$$P(\text{at least one head}) = \frac{n(E)}{n(S)} = \frac{3}{4}$$

(c) "At ~~least~~ most one head" means no heads or one head. ie TT, TH, HT.

Hence $n(E) = 3$.

So $P(\text{at most one head}) = \dfrac{n(E)}{n(S)} = \dfrac{3}{4}$

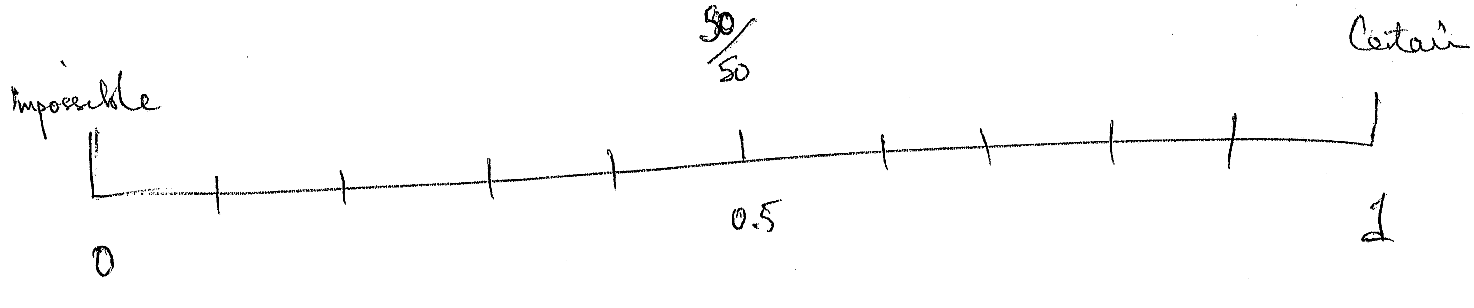
The probability of an event E will always be a number between and including zero and one.

ie $0 \leq P(E) \leq 1$.

When an event cannot occur (ie, contains no members in the sample space), the probability is zero.

When an event is certain to occur, the probability is one.

The sum of the probabilities of all the outcomes in the sample space will always be one.



For any event, E, we denote by $\bar{E}$ the event that E does not occur.

Hence, for any event E, $P(\bar{E}) = 1 - P(E)$.

e.g.

If a dice is rolled, the probability that 4 will not occur, symbolised by ~~ie 1,2,3,5,6.~~ $P(\bar{4})$, is $\frac{5}{6}$.

This shows that they are five ways that a 4 will not occur, ie 1,2,3,5,6.

$$
\begin{aligned}
\text{So} \quad P(\bar{4}) &= 1 - P(4) \\
&= 1 - \frac{1}{6} \\
&= \frac{5}{6}.
\end{aligned}
$$

Odds

Odds can be expressed as a fraction or ratio, e.g $\frac{5}{1}$ or $5:1$ or 5 to 1.

The formula for odds are

$$\text{odds in favor} = \frac{P(E)}{1 - P(E)}$$

$$\text{odds against} = \frac{P(\bar{E})}{1 - P(\bar{E})},$$

where $P(E)$ is the probability that event E occurs and $P(\bar{E})$ is the probability that the event E does not occur.

Question.

1. A card is drawn from a deck.

(a) Find the odds in favor of getting an ace

(b) Find the odds against getting an ace.

Solution

1. (a)

There are 4 aces in a deck of 52 cards.

$$P(A) = \frac{4}{52}.$$

Hence

$$\text{odds in favour} = \frac{P(A)}{1 - P(A)}$$

$$= \frac{\frac{4}{52}}{1 - \frac{4}{52}}$$

$$= \frac{\left(\frac{4}{52}\right)}{\left(\frac{48}{52}\right)} = \frac{4}{\cancel{52}} \times \frac{\cancel{52}}{48}$$

$$= \frac{4}{48}$$

$$= \frac{1}{12}$$

Hence, the odds in favor are 1 : 12.

(b) There are 48 ways of not getting an ace.

So $$P(\bar{A}) = 1 - P(A)$$

$$= 1 - \frac{4\cancel{8}}{52}$$

$$= \frac{48}{52}$$ ~~as exp~~.

odds against $$= \frac{\frac{48}{52}}{1 - \frac{48}{52}}$$

$$= \frac{\frac{48}{52}}{\left(\frac{4}{52}\right)} = \frac{48}{\cancel{52}} \times \frac{\cancel{52}}{4}$$

$$= \frac{12}{1}$$

Hence, the odds against an ace are $12:1$.

Definition.

If the odds in favour of an event E are $a:b$, then the probability that the event will occur is

$$P(E) = \frac{a}{a+b}$$

If the odds against an event E are $c:d$ then the probability that E will not occur is

$$P(\bar{E}) = \frac{c}{c+d}.$$

e.g. If Marlboro College are 3 to 1 favourites to beat Bennington at Basketball then

the probability of Marlboro College beating Bennington in a basketball game is

$$P(E) = \frac{3}{(1+3)}$$

$$= \frac{3}{4}$$

Two events are <u>mutually exclusive</u> if they cannot occur at the same time (ie they have no outcomes in common).

e.g.

the events drawing an ace and an 8 from a deck are mutually exclusive, since no card ~~the~~ can be both and ace and 8.

However, the events drawing an ace and a diamonds are not mutually exclusive, since one can select the ace of diamonds.

Addition rule 1

When two events are mutually exclusive, the probability that A or B will occur is

$$P(A \text{ or } B) = P(A) + P(B).$$

Addition rule 2

If events A and B are not mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

(b)

there are four kings and 13 clubs. However, the king of clubs has been counted twice. Hence king or clubs is not mutually exclusive.

$$P(\text{king or club}) = P(\text{king}) + P(\text{club}) - P(\text{king of clubs})$$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$$

$$= \frac{16}{52}$$

$$= \frac{4}{13}$$

Example

1. A single card is drawn from a deck

(a) find the probability that it is a club, diamond or heart?

(b) find the probability that it is a king or club?

Solution

(a) club, diamond or hearts are mutually exclusive events, where P(club) = P(hearts) = P(diamonds) $= \frac{13}{52}$

So P(heart or club or diamond) =

$$\frac{13}{52} + \frac{13}{52} + \frac{13}{52} = \frac{39}{52}$$

$$= \frac{3}{4}$$

Two events, A and B, are <u>independent</u> if the fact that A occurs does not affect the probability of B occurring.

Two events, A and B, are said to be <u>dependent</u> if the fact that A occurs does affect the ~~probab~~ (probability) outcome or occurrence of B.

eg. of independent events:—

- rolling a die and getting a 6, then rolling a second die and getting a 3.
- drawing a card from a deck and getting a queen, replacing it, and drawing a second card and getting a queen.

eg of dependent events:—

- drawing a card from a deck, not replacing it, and then drawing a second card.
- ~~It~~ The weather is cloudy, and then it rains.

Multiplication rule 1

when two events A and B are independent, the probability of both occuring is

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Multiplication Rule 2

When two events are dependent, the probability of both occuring is

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A),$$

where $P(B \mid A)$ (vertical line) is the probability that B occurs given that event A has occured.

$P(B \mid A)$ is the conditional probability of event B occuring given event A has occured.

Example

1. A die is rolled four times. Find the probability of getting

(a) four sixes.

(b) four odd numbers.

2. A box contains three red balls, two blue balls, and one white ball. A ball is selected at random and its colour noted. Then ball is not replaced and a second ball is selected and its colour noted. Find the probability that

(a) Both balls are blue

(b) A red ball is selected first, and a blue ball is selected second.

Solution

1. a) $P(4 \text{ sixes}) = P(\text{six}) \cdot P(\text{six}) \cdot P(\text{six}) \cdot P(\text{six})$

$$= \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$$

$$= \frac{1}{1296}$$

b) $P(4 \text{ odd numbers}) = P(\text{odd number}) \cdot P(\text{odd number}) \cdot P(\text{odd number}) \cdot P(\text{odd number})$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$= \frac{1}{16}.$$

2. (a) P(both balls blue) = P(first ball blue) . P(second ball blue | first ball Blue)

ie P(blue ball first and blue ball second)

$$= \frac{2}{6} \cdot \frac{1}{5} = \frac{2}{30}$$

$$= \frac{1}{15}$$

[not Mutually exclusive events)

(b) P(red ball first, then blue ball second) = P(red ball first) × P(blue ball second given that first is red)

$$= \frac{3}{6} \cdot \frac{2}{5}$$

$$= \frac{1}{5}$$

[Mutually exclusive events]