Overview — polynomial algebra

1 General strategy for solving polynomial equations

Our goal is to solve polynomial equations, such as, say, $x^3 - 11x^2 - 10x + 200 = 0$. In other words, we are looking for values of x that makes the polynomial zero. One fact will be the key to all our efforts to tackle such equations: a product is zero when one of the factors is zero. Thus our general strategy will be to write the polynomial as a product. In the above example, the polynomial $x^3 - 11x^2 - 10x + 200$ can be written as a product as follows: $(x+4)(x-5)(x-10)$; this is easy to check by multiplying out the parentheses, but how we found it in the first place shall remain a mystery for now. So we reduced the hard problem of finding the zeros of $x^3 - 11x^2 - 10x + 200$ to the easy problem of finding the zeros of $(x+4)(x-5)(x-10)$. Since $(x+4)(x-5)(x-10)$ can be zero only when one of its factors is zero we see that the values of x that make the polynomial zero are -4, 5 and 10. So now we have our general strategy charted out. However, implementation of this strategy is not always straightforward, as our example suggests—it will frequently require much ingenuity to find the factorization from the polynomial. For this reason we shall be much concerned with factorization techniques.

2 Factoring by algebraic rules

Sometimes we find factorizations by using good old algebraic rules such as these:

$$
\begin{aligned}
(x+y)^2 &= x^2 + 2xy + y^2 \\
(x-y)^2 &= x^2 - 2xy + y^2 \\
x^2 - y^2 &= (x+y)(x-y) \\
x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \\
x^3 - y^3 &= (x-y)(x^2 + xy + y^2)
\end{aligned}
$$

Example. By the second of these rules, the equation $x^2 - 6x + 9 = 0$ can be rewritten as $(x-3)^2 = 0$, so the only solution is $x = 3$. □

3 Quadratic equations

Quadratic equations $ax^2 + bx + c = 0$ can always be solved by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Ambitious readers could verify this formula and see that it agrees with our factorization program by checking that

$$\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = x^2 + \frac{b}{a}x + \frac{c}{a}.$$

Thus we have found the factorization of any quadratic equation once and for all, which, in principle, liberates us from the burden of inventiveness involved in finding the factorizations in particular cases. Since the formula is fairly cumbersome, however, factorization tricks will sometimes still be preferable in practice.

Example. For the quadratic equation $2x^2 - 4x - 16 = 0$ we have $a = 2$, $b = -4$ and $c = -16$. Therefore the solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{4^2 - 4 \cdot 2 \cdot (-16)}}{2 \cdot 2} = \frac{4 \pm \sqrt{144}}{4} = \frac{4 \pm 12}{4} = \frac{16}{4} \text{ and } \frac{-8}{4} = 4 \text{ and } -2 \quad \square$$

4 Factoring higher degree polynomials

4.1 Equations with no constant term

When all terms in the equation involve x, zero is obviously a solution. We then factor out x and study the solutions to the remaining expression.

Example. The equation $2x^3 - 4x^2 - 16x = 0$ has no constant term. Factoring out x yields $x\left(2x^2 - 4x - 16\right) = 0$. So zero is a solution and the other solutions will come from the other factor, $2x^2 - 4x - 16$, being zero. We saw above that $2x^2 - 4x - 16$ is zero when x is 4 or -2, so our original equation $2x^3 - 4x^2 - 16x = 0$ has the solutions 0, 4, and -2. $\square$

4.2 Equations involving only even powers

A polynomial in x with only even powers may also be regarded as a polynomial in x^2.

Example. The equation $x^4 - 6x^2 + 9 = 0$ can be attacked as a second degree equation by writing it as $(x^2)^2 - 6(x^2) + 9 = 0$, or, if we prefer, we can let u stand for x^2 and the equation becomes $u^2 - 6u + 9 = 0$. So x^2 must equal the solutions of the equation $u^2 - 6u + 9 = 0$. We solved this second degree equation above and found that the only solution was 3, so $x^2 = 3$, so $x = \pm\sqrt{3}$. $\square$

4.3 Equations with one obvious factor or solution

If $x = r$ is a solution to a polynomial equation then we know that $(x - r)$ is a factor of the polynomial. Therefore we can factor out $(x - r)$ from the polynomial and study the zeros of the remaining factor.

Example. The equation $x^3 - x^2 - x + 1 = 0$ has the obvious solution 1, so $(x - 1)$ must be a factor of $x^3 - x^2 - x + 1$. Then it is not hard to find the factorization $x^3 - x^2 - x + 1 = (x - 1)(x^2 - 1)$, so the equation has the two solutions 1 and -1. Instead of noticing the solution 1 we might have noticed the factor $(x - 1)$ and arrived at the factorization from there, perhaps through the intermediate step $x^3 - x^2 - x + 1 = (x - 1)(x^2) + (x - 1)(-1) = (x - 1)(x^2 - 1)$. $\square$

5 Division of polynomials

If $x = r$ is a solution to a polynomial equation then we know that $(x - r)$ is a factor of the polynomial and, as we saw above, factoring out $(x - r)$ will help us solve the equation. This is not always easy, however. It is one thing to know a solution, but to factor it out is a different matter. The situation is analogous to divisibility of ordinary integers. We know that 186327 is divisible by 9 since the sum of its digits is divisible by 9, but that doesn't help us find the value of $186327 \div 9$. For this we would need long division or something like it. We shall study the analog of long division for polynomials. It will help us factor out solutions $(x - r)$ from polynomials, just as ordinary long division helps us factor out the factor 9 from 186327.

5.1 The division algorithm for polynomials

Example. We consider again the equation $x^3 - x^2 - x + 1 = 0$, which we factored above by spotting the root $x = 1$ and using clever guesswork to factor out $(x - 1)$. Division of polynomials gives a systematic way of factoring out known factors. So the problem is to divide $x^3 - x^2 - x + 1$ by $x - 1$. We write this as follows.

$$\begin{array}{r|llll} \cline{2-5} x+1 & x^3 & -x^2 & -x & +1 \end{array}$$

As in ordinary long division, we need to determine how many times $x + 1$ fits into $x^3 - x^2 - x + 1$. When we are dealing with polynomials we should look only at the highest degree terms: the x from $x + 1$ and the x^3 from $x^3 - x^2 - x + 1$. The greatest "number of times" x goes into x^3 is x^2. Therefore we record x^2 as the first term of our solution as follows.

$$\begin{array}{r|llll} \multicolumn{1}{r}{} & & x^2 & & \\ \cline{2-5} x+1 & x^3 & -x^2 & -x & +1 \end{array}$$

So we found that we can take away x^2 number of $(x + 1)$'s from $x^3 - x^2 - x + 1$. Now we need to know what's left of $x^3 - x^2 - x + 1$, i.e. we need to calculate $x^3 - x^2 - x + 1 - (x^2)(x + 1)$. To do this we write the result of multiplying out $(x^2)(x + 1)$ under $x^3 - x^2 - x + 1$ and then subtract.

$$\begin{array}{r|llll} \multicolumn{1}{r}{} & & x^2 & & \\ \cline{2-5} x+1 & x^3 & -x^2 & -x & +1 \\ & x^3 & -x^2 & & \\ \cline{2-5} & & & -x & +1 \end{array}$$

Now we do the same thing again with what's left. Look at the highest degree terms of $x - 1$ and $-x + 1$: how many x's can we take away from $-x$? Or, in other words, what can we multiply x by to get $-x$? -1, of course. So we put down -1 as the next term in our solution and subtract $(x - 1)(-1)$ from $-x + 1$.

$$\begin{array}{r|llll} \multicolumn{1}{r}{} & & x^2 & & -1 \\ \cline{2-5} x+1 & x^3 & -x^2 & -x & +1 \\ & x^3 & -x^2 & & \\ \cline{2-5} & & & -x & +1 \\ & & & -x & +1 \\ \cline{2-5} & & & & 0 \end{array}$$

Now the remainder is zero, which means that when we have taken away $x^2 - 1$ number of $(x + 1)$'s from $x^3 - x^2 - x + 1$ we are left with nothing. In other words, $x^3 - x^2 - x + 1 = (x + 1)(x^2 - 1)$, and we have thus factored our polynomial, as desired. $\square$

5.2 Finding initial solutions

To make use of the division of polynomials algorithm for solving polynomial equations we need to know one solution of the equations so that we know what to factor out. When trying to find this initial solution the following fact is helpful: a rational solution to a polynomial equation must be of the form

$$\frac{\text{divisor of the constant term}}{\text{divisor of the coefficient of the highest degree term}}$$

Example. Consider the equation $2x^4 - 4x^3 - x^2 - x + 6 = 0$. The divisors of the constant term are 1, 2, 3 and 6. The divisors of the coefficient of the highest degree term 1 and 2. Thus the candidates for rational solutions of the equation are 1, 2, 3, 6, $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$ and $\frac{6}{2}$. Testing these in the original equation, we see that 2 is indeed a solution. □

6 Rational equations

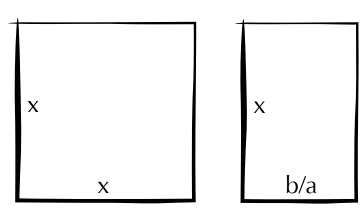
Rational equations can be solved by converting them to ordinary polynomial equations by multiplying by the denominators. (A technical note: This would not be allowed if one of the denominators was zero. The resulting polynomial equation, however, doesn't recall that it came from a rational equation, so it may display solutions that correspond to a denominator being zero in the original equation. Such solutions must of course be discarded.)

Example. Consider a rectangle with perimeter 2. Call the longer side x. Then the shorter side will be $1 - x$. Supposedly, the most aesthetically pleasing rectangle is the one where the shorter side is to the longer side as the longer side is to the sum of both sides, i.e. $\frac{1-x}{x} = \frac{x}{1}$. Multiplying up the denominators gives $1 - x = x^2$, which we may rewrite as $x^2 - x + 1 = 0$ and apply the quadratic formula to obtain the solutions $x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$. Apparently, there is a negative and a positive solution; if we wish to build ourselves one of those aesthetically perfect rectangles with perimeter 2 then we must clearly choose the positive root and make the longer side equal to the "golden ratio" $\frac{1+\sqrt{5}}{2}$. □

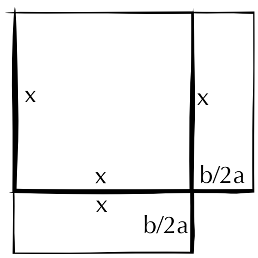
7 Bonus topics

7.1 Completing the square

We shall now derive the solution formula for quadratic equations by means of a useful method called completing the square. We wish to solve $ax^2 + bx + c = 0$. Let's divide by a and move the constant term to the other side to get $x^2 + \frac{b}{a}x = -\frac{c}{a}$. The left hand side can be interpreted geometrically as a square and a rectangle:



Cut the rectangle in half and click the pieces to the square:



Now we see how to "complete the square"—we must add the bottom right piece, which is a square with side $\frac{b}{2a}$. So we started with $x^2 + \frac{b}{a}x$ and saw that we needed to add $\left(\frac{b}{2a}\right)^2$ to turn it into one big square with side $x + \frac{b}{2a}$. So the left hand side of our equation is now $\left(x + \frac{b}{2a}\right)^2$. Since we added the little square $\left(\frac{b}{2a}\right)^2$ in the process, we must add it to the right hand side as well; so the right hand side is now $\left(\frac{b}{2a}\right)^2 - \frac{c}{a}$. That is,

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

Taking the square root of both sides we get

$$x + \frac{b}{2a} = \pm\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

or, solving for x,

$$x = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

or, putting the right hand side on the common denominator $2a$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

7.2 The geometric series

Applying the division algorithm to 1 divided by $1 - x$, where x is between 0 and 1, gives

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

This says for example that

$$\frac{1}{1 - \frac{1}{2}} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots,$$

i.e.

$$2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

Meditate on that for a while.