Combinations

Definition

A selection of objects without regard to order is called a combination.

E.g.

Given the letters A, B, C, D, list the permutations (arrangement of objects where order matters) and combinations of selecting two letters.

Solution

e.g. ~~Permutations~~

Ex

Suppose that a designer wishes to select three different colors of material to design a new dress, and the designer has five different colors. How many different possibilities can there be in this situation?

Solution

Permutations

AB	BA	CA	DA
AC	BC	CB	DB
AD	BD	CD	DC

Combinations

AB	BC
AC	BD
AD	CD

Note that the permutation AB is different from BA. However, in combinations, AB is the same as BA, and so one of the two is listed.

Solution

Let the colors be Red, Yellow ~~and~~, Orange, Green and Blue.

As the colors chosen have to be different, we find that there are $5 \times 4 \times 3 = 60$ choices of color if the order of the colors matter. However, the order of the chosen colors do not matter. For example, a selection of Red, Yellow and Green is the same thing as Yellow, Green and Red.

Each selection of 3 colors can be ordered in

3! ways.

e.g. Red, Yellow and Green (RYG)

can be ordered as:

RYG
RGY
YRG
YGR
GRY
GYR.

So the total number of combinations of Red, Yellow, Orange, Green and Blue into choices of 3 colors is

$$\frac{60}{6} = 10$$

ie.

RYG	RGO	ROB	YGO	YOB	GOB
RYO	RGB		YGB		
RYB					

penc

Combinations Rule

The number of combinations of r objects selected from n objects is denoted by nC_r [or $C(n,r)$] and is given by the formula

$$nC_r = \frac{n!}{(n-r)!\,r!}$$

Please recall that $n!$ is called "n factorial", where $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ (n is a positive integer)

So $0! = 1$ (defined)

$1! = 1$

$2! = 2\cdot 1 = 2$

$3! = 3\cdot 2\cdot 1 = 6$

$4! = 4\cdot 3\cdot 2\cdot 1 = 24$

$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120.$

We have seen that

$n=4,\ r=2.$

$$_4C_2 = \frac{4!}{2!2!} = 6.$$

(letter combinations of A, B, C, D into pairs).

We have also seen that

$n=5,\ r=3.$

$$_5C_3 = \frac{5!}{2!\,3!} = 10$$

(picking 3 color combinations from 5 colors).

Question

1. How many different ways can five cards be selected from a standard deck of 52 cards.

2. Harry has to visit 10 cities. He can visit any three in one day. How many different ways can he select three cities? Assume that distance is not a factor.

Solution

1.

$$ {}_{52}C_5 = \frac{52!}{(52-5)! \times 5!} $$

$$ = \frac{52!}{47! \times 5!} $$

$$ = 2{,}598{,}960 $$

2.

$$ {}_{10}C_3 = \frac{10!}{(10-3)! \times 3!} $$

$$ = \frac{10!}{7!\,3!} = 120 $$

Question

3. In a club there are 7 women and 5 men. A committee of 3 women and 2 men are to be chosen. How many different possibilites are there?

4. A committee of 5 people must be selected from 5 men and 8 women. How many different ways can the selection be done if there are to be at least 3 women on the committee.

Solution

3. We must select 3 women from 7 women, which is done in $7C_3$ ways.

Next, we must select 2 men from 5 men, which is done in $5C_2$ ways.

We can now put the groups of 3 women and of 2 men together in any way. (ie ~~Counting rul~~ Fundamental Counting Rule)

Total number of ways.

$$7C_3 \times 5C_2 = 35 \times 10$$

$$= 350.$$

Solution

4.

A committee of at least 3 women means that the committee can consist of :-

- 3 women and 2 men, or
- 4 women and 1 man, or
- 5 women.

There are 5 men and 8 women.

Hence number of combinations of

women ↓ men ↓

- 3 women and 2 men : $_8C_3 \cdot {_5C_2}$
- 4 women and 1 man : $_8C_4 \cdot {_5C_1}$
- 5 women (and 0 men): $_8C_5$ (note that $_5C_0 = 1$)

$$_8C_3 \cdot {}_5C_2 = \frac{8!}{5!\,3!} \cdot \frac{5!}{3!\,2!}$$

$$= 56 \cdot 10$$

$$_8C_4 \cdot {}_5C_1 = \frac{8!}{4!\,4!} \cdot \frac{5!}{4!\,1!}$$

$$= 70 \cdot 5$$

$$_8C_5 = \frac{8!}{3!\,5!}$$

$$= 56$$

Total number of committees :

$56 \times 10 + 70 \times 5 + 56 = 966.$

Questions

5. On an exam, a student must select two essay questions from six questions, and 10 multiple choice questions from 20 multiple choice questions to answer. How many different ways can the student select questions to answer?

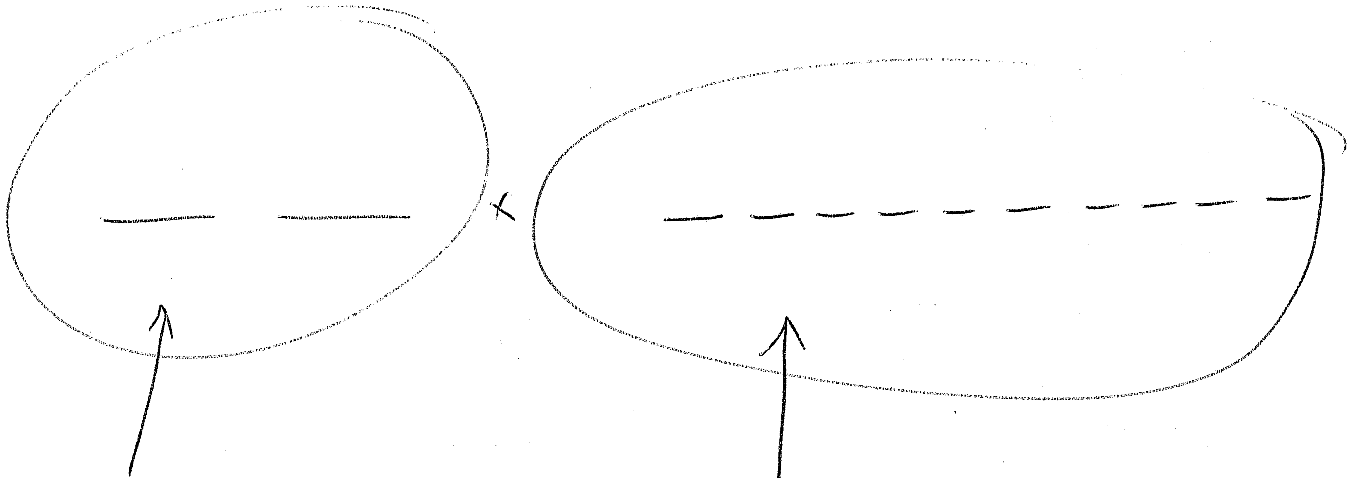
6. A chemist has 9 samples of solution, of which 4 are type A and 5 are type B. If the chemist chooses 3 of the solutions at random, determine the number of ways

a) the chemist can have exactly <u>one</u> type A solution

b) the chemist can have <u>at least</u> one type A solution.

Solution

5.



$${}_{6}C_{2} \times {}_{20}C_{10}$$

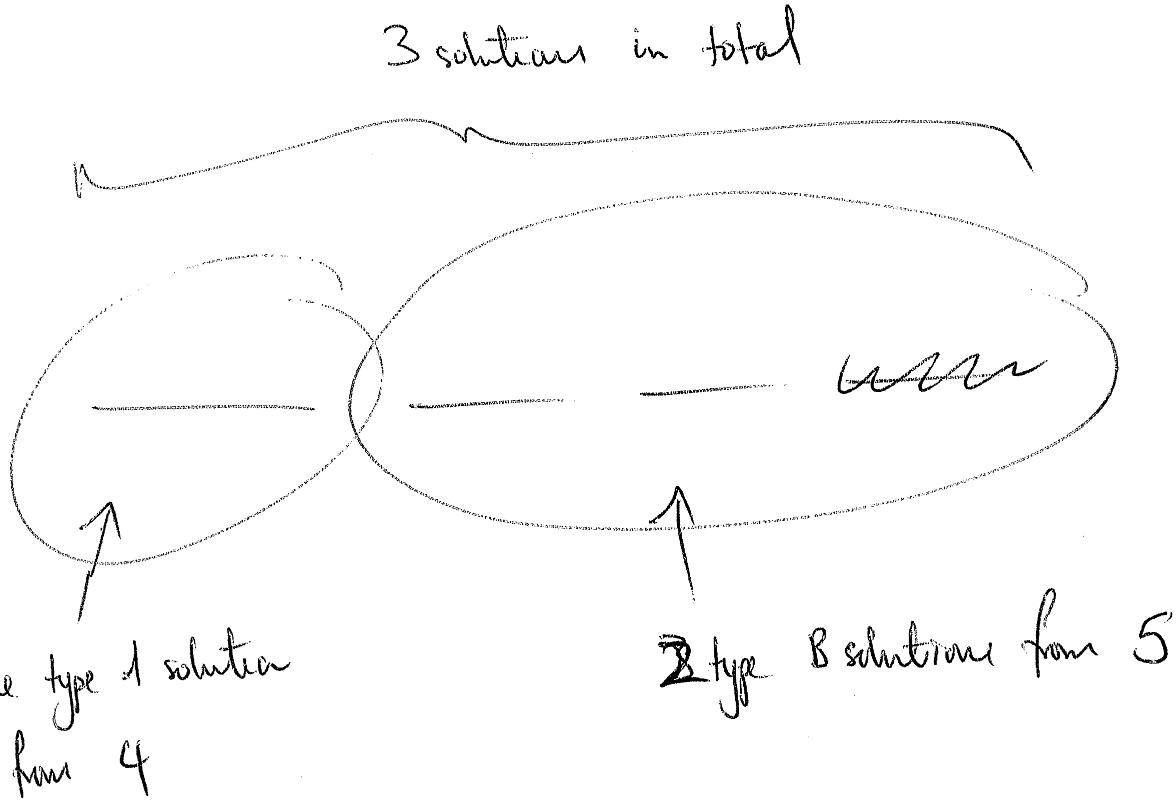
$$= \frac{6!}{(6-2)!\,2!} \times \frac{20!}{(20-10)!\,10!}$$

$$= \frac{6!}{4!\,2!} \times \frac{20!}{10!\,10!}$$

$$= 2,771,340$$

6.

(a)



$$= {}_4C_1 \times {}_5C_2$$

$$= \frac{4!}{(4-1)! \times 1!} \times \frac{5!}{(5-2)!\,2!}$$

$$= \frac{4!}{3!\,1!} \times \frac{5!}{3!\,2!}$$

$$= 4 \times 10$$

$$= 40$$

(b) We have 3 solutions chosen at random. These choices of solution fall into the following categories:-

- exactly one type A solution and exactly 2 type B solutions.
- exactly 2 type A solutions and exactly 1 type B solution
- exactly 3 type A solutions.

from part (a) 40 in total

$_4C_2 \times {_5C_1} = 6 \times 5 = 30$

$_4C_3 = 4$

Total for at least one type A solution : $40 + 30 + 4 = 74$

(b) Another method:

Picking any 3 solutions at random fall into the following categories :-

- exactly 1 type A solution and exactly 2 type B solutions)
- exactly 2 type A solutions and exactly 1 type B solution
- exactly 3 type A solutions (0 type B solutions)
- exactly 3 type B solutions (0 type A solutions).

Picking any 3 solutions at random

$$9C_3 = \frac{9!}{6!\,3!} = 84$$

Picking 3 type B solutions

$$5C_2 = \frac{5!}{3!\,2!} = 10$$

$$\overline{74}$$