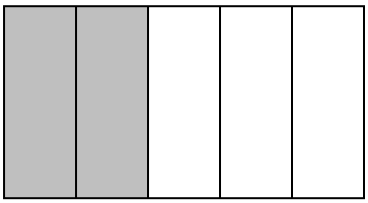
Arithmetic of Fractions

The rule for adding two fractions with the same denominator is $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$. This rule is obvious when you read it out loud; for example,

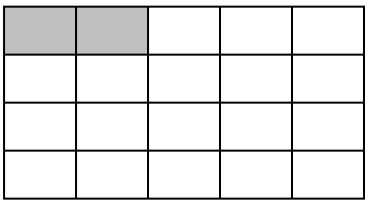
$$\frac{1}{3} + \frac{4}{3} = \frac{1+4}{3} \qquad \text{“one third plus four thirds = one plus four thirds”}$$

Several other rules also become clear when read aloud, such as the corresponding rule for subtraction and the rule $a \cdot \frac{1}{b} = \frac{a}{b}$ (e.g., “two times one third = two thirds”), which says that to multiply by $\frac{1}{b}$ is the same thing as to divide by b. Alas, linguistics is not as powerful as mathematics so we soon have to start doing some actual work. Let’s say for instance that we wish to add two fractions with different denominators. Then what? The trick is to reduce this problem to the known case above. We do this by the following rule: $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$. That is: multiplying the numerator and the denominator with the same number does not change the value of the fraction. This is true because if you are sharing a pizza with a friend then it doesn’t matter if you cut it into two pieces and take one half each or if you cut it into four pieces and take two quarters each, so $\frac{1}{2} = \frac{1 \cdot 2}{2 \cdot 2} = \frac{2}{4}$, and so on. (Of course one can also read $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ from right to left: dividing the numerator and the denominator with the same number does not change the value of the fraction.) Now we can add any fractions by first making their denominators equal; e.g., $\frac{1}{2} + \frac{3}{8} = \frac{4}{8} + \frac{3}{8} = \frac{7}{8}$.

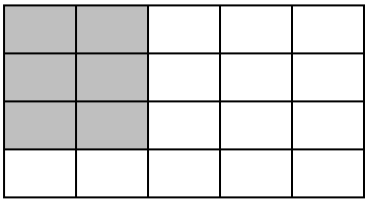
Next: multiplication. As we know, multiplying by $\frac{a}{b}$ is the same thing as dividing by b and multiplying by a. So what is the result of multiplying two fractions, say $\frac{2}{5}$ and $\frac{3}{4}$? Start with $\frac{2}{5}$:



Now divide by 4:



Now multiply by 3:



Thus we see that the result of $\frac{2}{5} \cdot \frac{3}{4}$ is $\frac{6}{20}$. And in general: $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$.

Only division remains. Actually, this is not very exciting at all. Division by $\frac{a}{b}$ is just multiplication by $\frac{b}{a}$, as we can see by using $\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$ twice:

$$\frac{\frac{c}{d}}{\frac{a}{b}} = \frac{\frac{c}{d} \cdot b}{\frac{a}{b} \cdot b} = \frac{\frac{c}{d} \cdot b}{a} = \frac{\frac{c}{d} \cdot b \cdot \frac{1}{a}}{a \cdot \frac{1}{a}} = \frac{c}{d} \cdot \frac{b}{a}.$$