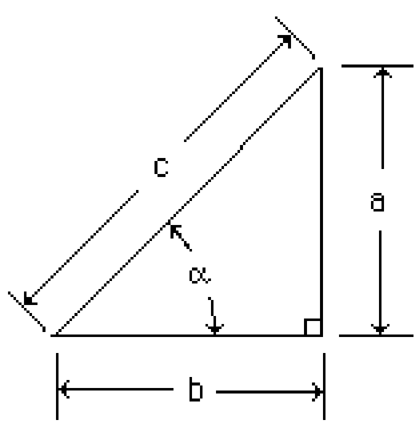
# More About Trigonometric Functions

EMLS 33

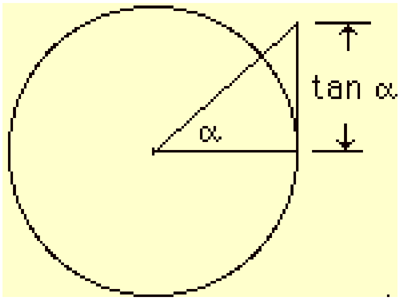
Objectives: To learn about tangent, cotangent, secant and cosecant functions.
**Recall:**

1. There are four other trigonometric functions that naturally arise when angles become part of a mathematical problem. If we consider the right triangle below to have sides of length $a$, $b$ and $c$, then we make the following definitions:

   - Tangent = $\tan(\alpha) = \frac{a}{b}$
   - Cotangent = $\cot(\alpha) = \frac{b}{a}$
   - Secant = $\sec(\alpha) = \frac{c}{b}$
   - Cosecant = $\csc(\alpha) = \frac{c}{a}$




2. A geometric measure of the $\tan a$ can be represented by the height of a right triangle having base angle $a$ and base 1. This is the height of a triangle with base angle $a$ at the center of a circle of radius 1. See illustration below.



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3. Several important relations are displayed below.

   $\tan a = \frac{\sin a}{\cos a}$

   $\cot a = \frac{\cos a}{\sin a} = \frac{1}{\tan a}$

   $\sec a = \frac{1}{\cos a}$

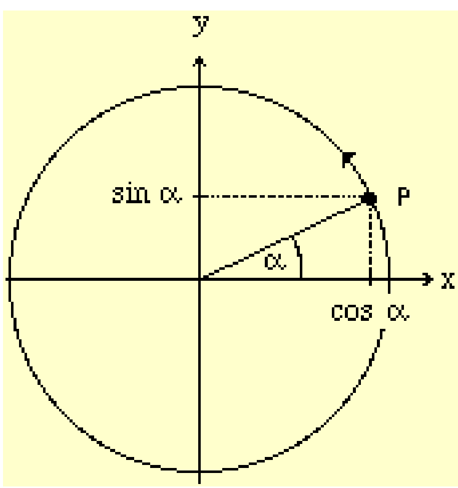
   $\csc a = \frac{1}{\sin a}$

4. We recall that the Pythagorean Theorem says that $(\sin a)^2 + (\cos a)^2 = 1$. From this, we deduce that

$$1 + (\tan a)^2 = (\sec a)^2$$

   and

$$1 + (\cot a)^2 = (\csc a)^2$$



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5. You will often find that solutions to trigonometric calculations simplify through normal algebraic cancellation laws. For example, the expression

$$\frac{\sec a}{\csc a} \tan a + 1$$

   can be written as

$$\frac{\frac{1}{\cos a}}{\frac{1}{\sin a}} \cdot \frac{\sin a}{\cos a} + 1 \quad = \quad \frac{1}{\cos a} \cdot \frac{\sin a}{1} \cdot \frac{\sin a}{\cos a} + 1$$

$$
\begin{aligned}
&= \frac{\sin a}{\cos a} \cdot \frac{\sin a}{\cos a} + 1 \\
&= \frac{\sin^2 a}{\cos^2 a} + 1 \\
&= \frac{\sin^2 a + \cos^2 a}{\cos^2 a} \\
&= \frac{1}{\cos^2 a} \\
&= \sec^2 a
\end{aligned}
$$

6. In practical problems you may have to solve equations where the unknown is a trigonometric function. For example, the equation may turn out to be something like $(\cos a)^2 = \cos a$. How do you solve such an equation when $0 \leq a \leq 2\pi$?

   - Set $(\cos a)^2 - \cos a = 0$.
   - Then $\cos a(\cos a - 1) = 0$.
   - This tells us that either $\cos a = 0$ or $\cos a = 1$.
   - The first possibility leads to $a = \pi/2$ or $a = 3\pi/2$ while the second leads to $a = 0$ or $a = 2\pi$. Thus, the solutions are $0$, $\pi/2$, $3\pi/2$, and $2\pi$.

**Examples**

1. Show that

$$\frac{\cos \alpha}{1 - \sin \alpha} - \tan \alpha = \sec \alpha.$$

   *Solution:* Work with the more complicated side. In this case the left side is the complicated side. We have

$$
\begin{aligned}
\frac{\cos \alpha}{1 - \sin \alpha} - \tan \alpha &= \frac{\cos \alpha}{1 - \sin \alpha} - \frac{\sin \alpha}{\cos \alpha} \\
&= \frac{\cos^2 \alpha - \sin \alpha + \sin^2 \alpha}{(1 - \sin \alpha) \cos \alpha} \\
&= \frac{1}{\cos \alpha}
\end{aligned}
$$

2. Solve the equation $(\sec x)^2 = 2 \tan x$ for $-\pi/2 \leq x \leq \pi/2$.

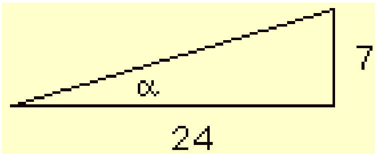
   *Solution:* Subtract $2 \tan x$ from both sides and use the identity $1 + \tan^2 x = \sec^2 x$ to get

$$(1 + \tan^2 x) - 2tanx = 0$$

   . Then let $y = \tan x$ to get $y^2 - 2y + 1 = 0$. Solve this quadratic equation to get $y = 1$. Hence $\tan x = 1$. Since $-\pi/2 \leq x \leq \pi/2$, we know that $x$ must be $\pi/4$.

3. Suppose that you know that $\tan a = \frac{7}{24}$ and that $0 \leq a \leq \pi/2$. Find $\sin a$ and $cosa$.

*Solution:* Use the trigonometric identity $1 + \tan^2 x = \sec^2 x$ to find $\sec a$ and use it to get $\cos a$. First draw a triangle like the one below.




Use the Pythagorean theorem to find the length of the hypotenuse of this triangle:

$$c^2 = 24^2 + 7^2 = 576 + 49 = 625.$$

Therefore $c = \sqrt{(625)} = 25$. Now you can find $\sin a$ and $\cos a$.

$\sin a = \dfrac{\text{opposite}}{\text{hypotenuse}} = \frac{7}{25}$

$\cos a = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \frac{24}{25}$.

# 1 Exercises

1. If $\cos a = 2/3$, what is $\sin a$?

2. If $\cos a = 4/5$, what is $\tan a$?

3. A right triangle has hypotenuse 5, base 4 and the angle between the base and the hypotenuse is $a$. Find $\tan a$, $\cot a$, $\sec a$ and $\csc a$.

4. A right triangle has a base of size 4 and height of size 5. Find the values of all the trigonometric functions of the base angle.

5. A line joining the coordinates $(0, 0)$ with $(3, -4)$ makes an angle $a$ with the horizontal line through $(0, 0)$. Find $\tan a$, $\cot a$, $\sec a$ and $\csc a$.

6. A balloon is at an altitude of 200 feet above sea level and directly above the edge of a cliff. An observer on a beach sites the balloon through a telescope aimed at a 30 degree angle from the horizontal. How far is the balloon from the observer?

7. A surveying team measures an angle of ?? radians from a point $A$ on a beach to a point $B$ at the top of a cliff. If point $A$ is 200 meters from the base of the cliff, how high is the cliff?

8. A crane is 80 feet long. For safety reasons it should never be lowered less than 55° from the horizontal. How high will the crane reach?

9. A tower casts a 120 foot shadow on horizontal ground from the sun when it makes an angle of 60 degrees. How tall is the tower?

10. Simplify $\sin x - \cos^2 x \sin x$.

11. Simplify $\tan^2 x \csc^2 x \cot^2 x \sin^2 x$.

12. Solve the equation $2\sin^2 x + 3\cos x = 0$ for $0 \leq x \leq \pi$.