Graphing Linear Functions

EMLS 25

1 Objectives

- To recall how to graph straight lines or linear functions.
- To learn to find x and y intercepts.
- To learn to find a line knowing a point and its slope.
- To recognize equations of vertical or horizontal lines.
- **Recall 1:** (Definition of a graph)

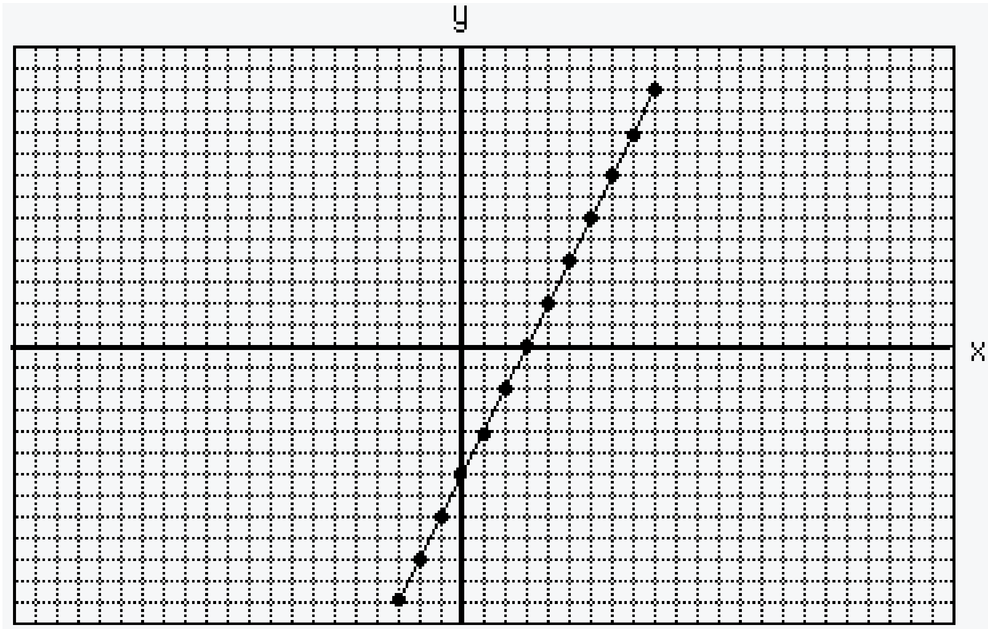
  Suppose that $f\colon A \to B$ is a function from A to B. The graph of f is the set of points in the plane described by $(x, f(x))$, where x is in the domain of f. If we plot $(x, f(x))$ for a sufficient number of values of x, we find that the points $(x, f(x))$, plotted in the Cartesian plane, are ordered along a curve.

  In this Workout, we shall be looking at graphs of functions that come from linear expressions, that is where the variable x is not raised to a power. For example, $f(x) = 2x - 6$.

  The graph of such an expression will be a set of points $(x, f(x))$ that are ordered along a straight line. Indeed, if we look at a table of values for $(x, f(x))$, we find

x	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
f(x)	-12	-10	-8	-6	-4	-2	0	2	4	6	8	10	12

  If we plot these points on the plane, and connect the points by straight lines, we find that all the points lie on the single straight line illustrated below.



Note: Since graphs of functions are plotted with reference to an x-axis and a y-axis, it is often useful to label the function y. We say that $y = f(x)$, or that y is a function of x. Notice that the value of $f(x)$ is measured along the vertical y axis. For the same reason, we may write equations, such as $y = 2x - 6$, when we mean $f(x) = 2x - 6$.

- **Recall 2:** (The x and y intercepts of a straight line.)

  Notice that any straight line that you can draw in the plane must intersect at least one of the axes, either the x-axis or the y-axis. Therefore, straight lines can be classified into three categories:

  - Lines that intersect only the x-axis.
  - Lines that intersect only the y-axis.
  - Lines that intersect both the x-axis and the y-axis.

  Lines of type 1 cannot be the graph of any function of y, since they are be vertical lines. Such lines are completely determined by where they cross the x-axis. For example if a vertical line crosses the x-axis at $x = 9$, then the x value of its graph must always be 9. In other words, the graph must be the set of all points $(9, y)$, and it does not matter what y is (y can be any real number.) Recall that a function must have a unique value of $f(x)$ for each value of x.

  Lines of type 2 are horizontal lines. Therefore they have the same y value for every x value. Thus, these lines are graphs of functions that take on the same value for all values of the independent variable x. Points on these lines have the form (x, b), where b is the value at which the line crosses the y-axis. For example, a horizontal line of height 5 is the set of points of the form $(x, 5)$ and is the graph of the function

$f(x) = 5$. A function for which the value of the function is the same for every value of x is called a *constant* function. The graph of any constant function is always horizontal and it crosses the y-axis at the value of the constant.

Lines of type 3 intersect the x-axis exactly once and intersect the y-axis exactly once. The point at which the line intersects the x-axis is called the *x-intercept*. The point at which the line intersects the y axis is called the *y-intercept.*

How do we find the intercepts?

Set $y = 0$ to find the x-intercept; set $x = 0$ to find the y-intercept.

- **Recall 3.**(The x- and y-intercepts determine the line)

  Two points determine a unique line. We can see this geometrically. With two points, we are able to draw an unambiguous, well-defined, unique line connecting those two points and extending indefinitely into space. How do we determine a line of type 3? The easiest two points to find are the x-intercept and the y-intercept. These points will help us graph the line, if we already know the function. Later we will learn how to find the function, if we already know the points.

  The x-intercept is a point of the form $(a, 0)$, where a is some fixed real number. The y-intercept is a point of the form $(0, b)$, where b is some fixed real number. These two points can easily be obtained from the function by setting y equal to 0 and x equal to 0, respectively. For example, consider the function is $f(x) = 2x - 6$. By setting $y = f(x) = 0$, we have $0 = 2x - 6$. Solving for x gives $x = 3$. So the x-intercept for the graph of this function is the point $(3, 0)$. Next, we set $x = 0$ to find the y-intercept. Doing so yields $(0, -6)$. We therefore have two points on our graph, $(3, 0)$ and $(0, -6)$. Draw the line connecting these two points to sketch the graph of $f(x) = 2x - 6$. Notice that we get the same straight line that appears in Recall 1; this time we simply plotted two points instead of 13.

- **Recall 4.** (The slope of a line)

  If you take another look at the line illustrated in Recall 1 (the graph of $f(x) = 2x-6$) you may notice that the y value increases by 2 units every time the x value increases by 1 unit. All lines of type 3 have the property that their y value increases or decreases each time the x value increases by one unit. The amount that the y value increases or decreases for each unit of increase of the x value is called the *slope* of the line. The line given by the function $f(x) = 2x - 6$ has a slope of 2. That is, y increases by 2 units every time x increases by 1 unit. To see this, take the point $(0, -6)$. By increasing x one unit you come to the point on the line given by $(1, -4)$. The difference between -4 and -6 is 2 units. The point (x, y) on the line has moved up 2 units.

  It is no accident that the coefficient of the x value of the function $f(x) = 2x - 6$ is the same as the slope of the line. In general the graph of a function

  $$y = ax + b$$

will be a straight line of slope a. The form of the function above is called the standard form of a line function. In this form a is the slope and b is the y-intercept of the line.

Another way of finding the slope is to pick any two points on the graph, say (x_0, y_0) and (x_1, y_1). Then the difference between the abscissa values x_1 and x_0 is the horizontal change, called the run. The difference between the ordinate values y_1 and y_0 is the vertical change, called the rise. The slope is then calculated as the ratio

$$\frac{\text{rise}}{\text{run}} = \frac{y_1 - y_0}{x_1 - x_0}$$

For example, if you picked the x-intercept $(3, 0)$ and the y-intercept $(0, -6)$ as the two points, then

$$\frac{\text{rise}}{\text{run}} = \frac{0 - (-6)}{3 - 0} = 2,$$

as expected.

- **Recall 5.** (The y-intercept)

  The y-intercept has a special use in determining the function from the graph. If the y-intercept and the slope are known, then the function is also known. All linear functions may be written as

  $$f(x) = ax + b,$$

  where a is the slope of the graph of $f(x)$ (as we have seen). What is the significance of the constant term b? Try setting x equal to 0 and notice that $f(0) = b$. Therefore, b is the value of y at which the line crosses the y axis. In other words, b is the y value of the y-intercept. This means that by knowing two quantities–the slope a and the y-intercept b–we may write the function that describes the line. For example, if a line has a slope of 2 and a y-intercept of $(0, -6)$, then the line is the graph of the function $f(x) = 2x - 6$.

- **Recall 6.**(How to find the line, knowing the slope and a point)

  The slope of a line may be computed knowing any two points (x_0, y_0) and (x_1, y_1). We have seen that slope m is given by the ratio

  $$m = \frac{y_1 - y_0}{x_1 - x_0} \tag{1}$$

  Suppose we know the slope $m = 3$ and a point $(3, 4)$ of a function $f(x)$, but want to find the rest of the function. Plugging what we do know into Equation 1 gives

  $$3 = \frac{y - 4}{x - 3},$$

  where x and y are unknown variables. We can rearrange this equation to be

  $$(y - 4) = 3(x - 3).$$

Any x and y values that make this equation true will be on the graph of $f(x)$ (can you see why?). In the same way, we can rearrange Equation 1 to give:

$$(y - y_0) \quad = \quad m(x - x_0). \tag{2}$$

This is the equation of the line with slope m and passing through the point (x_0, y_0). We call Equation 2 the *point-slope form* of a line.

- **Recall 7.** (How to find the line knowing only two points)

  If we know two points (x_0, y_0) and (x_1, y_1), then we can find the slope of the line through these two points by

  $$m = \frac{(y_1 - y_0)}{(x_1 - x_0)}.$$

  From the last Recall, we know how to find the equation of the line by knowing the slope and one point. Now we know the slope and the point (x_0, y_0), and we may proceed as we did in Recall 6.

2 Examples

1. Find the x- and y-intercepts of the graph of the function $f(x) = 4x + 12$.

   Solution: The y-intercept is the point $(0, f(0))$. Now, $f(0) = 4(0) + 12 = 12$, so the y-intercept is $(0, 12)$.

   The x-intercept is the point $(x, 0)$. Setting $f(x) = 0$ gives $0 = 4x + 12$, so $x = -3$. The x-intercept is therefore $(-3, 0)$.

2. Find the slope of the function $f(x) = -4x - 3$.

   Solution: We know that any function that can be put in the form

   $$f(x) = mx + b$$

   must have a graph that is a straight line with slope m. Therefore, the slope of $f(x)$ is $m = -4$.

   Note To see that the line with the equation given by the function $f(x) = mx + b$ has slope m, check the rise over run. Consider $(x, f(x))$ and $(x + 1, f(x + 1))$ and find

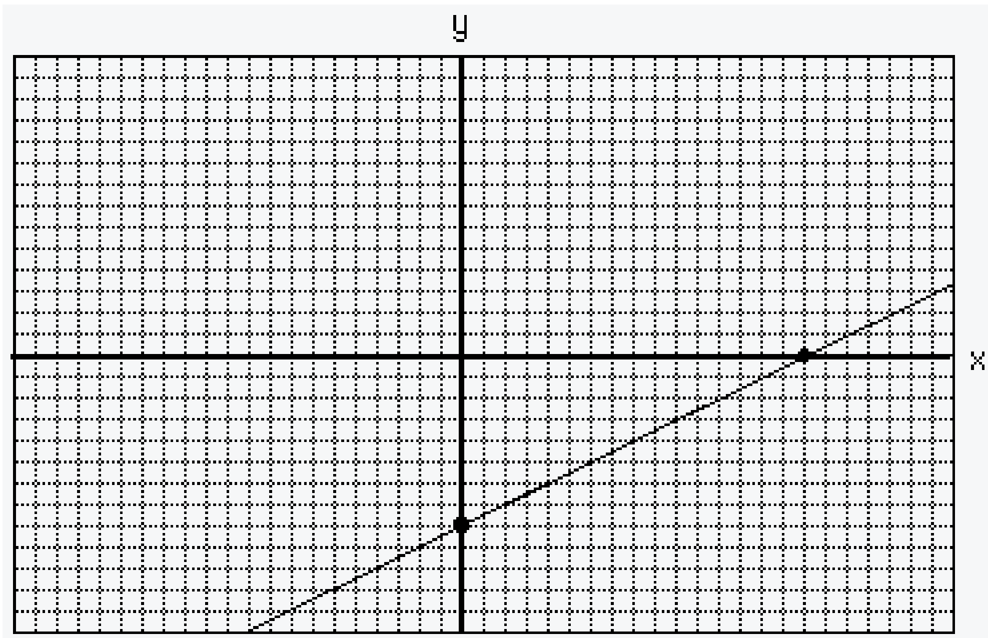
   $$\frac{\text{rise}}{\text{run}} = \frac{f(x+1) - f(x)}{(x+1) - (x)} = \frac{(m(x+1) + b) - (mx + b)}{1} = m.$$

3. Sketch the graph of the function $f(x) = \frac{(x-16)}{2}$.

   Solution: Rewrite this function in standard form as

   $$f(x) = x/2 - 8.$$

Notice that degree of this function (the highest power) is 1. Therefore, the graph of this function is a straight line. Compare the form of this function to the form $f(x) = mx+b$, where m is the slope of the line and b is the y-intercept. We find that $m = 1/2$ and $b = -8$. Knowing that the slope is $1/2$ and that the y-intercept is -8, we sketch the line below. We find the x-intercept solving the equation $x/2 - 8 = 0$. We get the point $(16, 0)$. So the graph is the line that goes through the points $(0, -8)$ and $(16, 0)$:



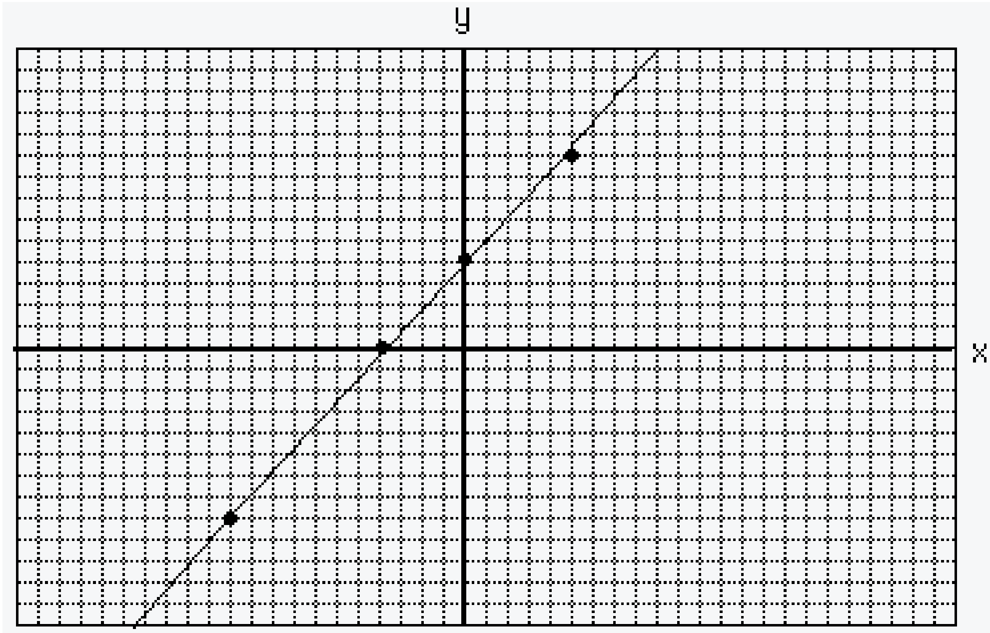
4. Find the function whose graph is the straight line passing through the two points $(-12, -8)$ and $(5, 9)$. Where does this line cross the y- axis? Where does it cross the x-axis?. Sketch the line through the two points $(-12, -8)$ and $(5, 9)$ to be sure that it does cross the axes at the computed x and y intercepts.

   Solution: First let's find the slope with the formula given in Recall 4. We get $m = 1$. Now we have a slope and a point, namely slope 1 and point $(5, 9)$ (or $(-12, -8)$). So the equation of the line is

$$y - 9 = 1(x - 5)$$

   . Hence, $y = x + 4$. To complete the example, notice that it crosses the y-axis at $(0, 4)$ (the y-intercept). It crosses the x-axis when $y = 0$, or at $(-4, 0)$ (the x-intercept). Notice that by comparing the form of the equation with the form of a function whose graph is the line with slope m and y-intercept b, we may check that our result is correct.

$$y = 1x + 4$$
$$f(x) = mx + b$$



3 Exercises

1. Sketch the graph of $6(x+3) = 2y-4$ by plotting the points $(-4, f(-4))$, $(-3, f(-3))$, $(-2, f(-2))$, $(-1, f(-1))$, $(0, f(0))$, $(1, f(1))$, $(2, f(2))$, $(3, f(3))$, and $(4, f(4))$.

2. Find the equation of the line that has slope 3 and y-intercept -8.

3. Put the equation $6(x + 5) = 2y - 4$ into the standard form of an equation of a straight line.

4. Find the equation of the line that has slope -2 and y-intercept -6.

5. Find the equation of the line that has slope -3 and passes through the point $(-4, 9)$.

6. Find the equation for the line with x-intercept at $(4, 0)$ and y-intercept at $(0, -9)$.

7. Find the x and y intercepts of the line $y = 4x + 28$.

8. Find the equation of the line passing through $(-3, 8)$ and $(5, 4)$.

9. Find the equation of the line passing through $(5, 3)$ and $(-7, -4)$.

10. Sketch the graph of the function $f(x) = 2x + 7$.

11. Use the same graph paper as you used in exercise 10 to sketch the graph of the function $g(x) = -2x + 7$.

12. Use the same graph paper as you used in exercise 10 to sketch the graph of the function $h(x) = -2x - 7$.

13. Use the same graph paper as you used in exercise 10 to sketch the graph of the function $s(x) = x$.

14. Use the same graph paper as you used in exercise 10 to sketch the graph of the function $t(x) = x + 7$.

15. Sketch the graph of the function $f(x) = 8$.

16. Sketch the line $x = 8$.

4 Solutions

2. $y = 3x - 8$
3. $y = 3x + 11$
4. $y = -2x - 6$
5. $y = -3x - 3$
6. $y = x - 9$
7. $(-7, 0)$ and $(0, 28)$