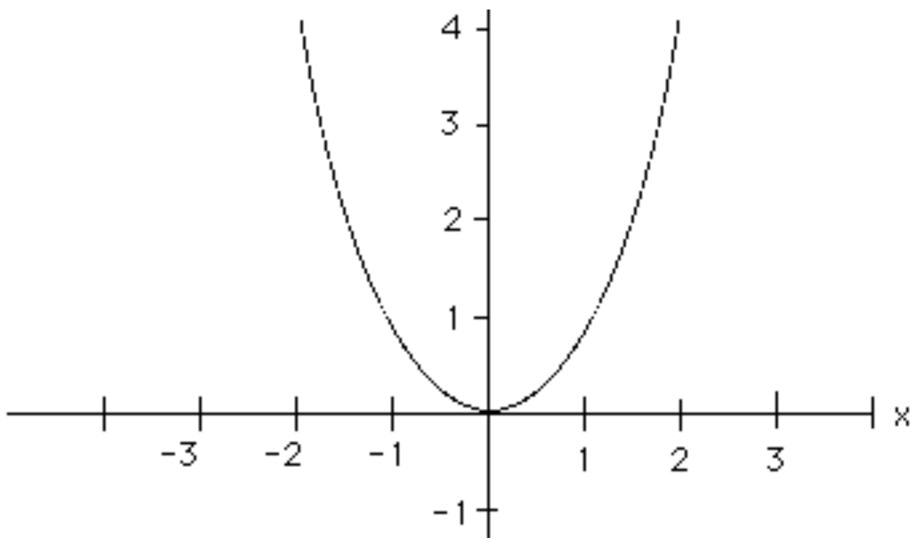
Graphing quadratic functions

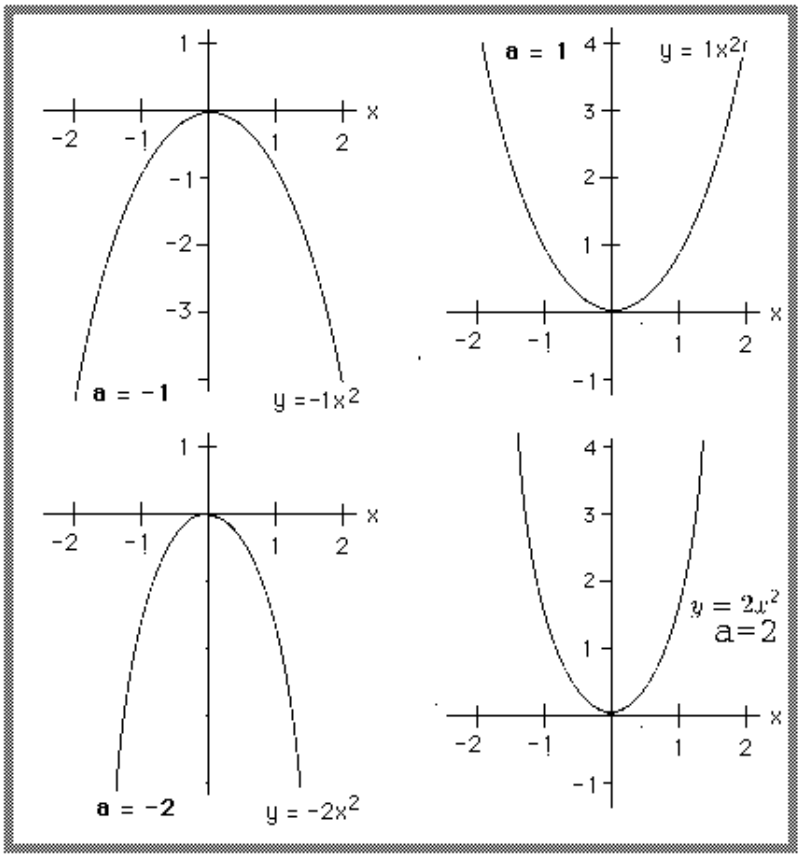
In the past we put quadratic equations in the form $y = ax^2 + bx + c$. This makes the equation look nice and become more readable. For the purposes of graphing, we would rather alter this form slightly. Recall that we represented lines in standard form as $y = mx + b$. Then we were able to read the critical information about slope and y-intercept directly from the form of the equation. The slope is m and the y-intercept is b. We wish to do the same with the graphs of quadratic functions. We will use several ideas that were touched on in past to provide a scheme for knowing how to graph a quadratic function.

1 The standard parabola centered at the origin

If you sketch the graph of the quadratic equation $y = x^2$, you find a curve that looks like the curve below.



This curve is called a parabola. The first thing to notice about a parabola is its symmetry. If you flip this parabola about the y-axis, you get the exact same parabola. Every parabola has such a line of symmetry. In the case of the parabola $y = x^2$, the y-axis is the line of symmetry. The point at which the parabola meets its line of symmetry is called the vertex of the parabola. Now, the basic shape and orientation will be determined by the coefficients of the quadratic $y = ax^2 + bx + c$. We take one possibility at a time. Suppose that the coefficients b and c are both 0. In this case our parabola is given by the equation $y = ax^2$. The question should now be: What role does the coefficient a play in orienting or shaping the graph? To find out, let a be a few different values and compare the values of a with the corresponding shapes of the sketches that we get. The sketches below suggest the role of a.



If the coefficient is negative, the parabola faces downward. If the coefficient is positive, then the parabola faces upward. The absolute value of the coefficient determines the sharpness of the curve at the vertex—the higher the absolute value, the steeper and sharper the parabola.

2 The general case $y = ax^2 + bx + c$

If we can put the equation into the form $y = a(x - h)^2 + k$, where h and k are specific real numbers then we would know the orientation and position of the parabola that is the graph of the quadratic equation. It turns out that this parabola has a vertex at the point (h, k). Again, the steepness of the parabola, or the sharpness at the vertex is determined by the coefficient a–the larger its absolute value, the steeper the parabola. For example, the curve $y = 2(x - 5)^2 + 7$ is a parabola opening upward with vertex at $(5, 7)$. The line of symmetry in this case is the vertical line $x = 5$.

3 Changing $y = ax^2 + bx + c$ to the form$y = a(x - h)^2 + k$

Any quadratic equation of the form $y = ax^2 + bx + c$ can be changed to the form $y = a(x - h)^2 + k$, if the proper values of h and k can be found. The process of finding such h and k is called completing the square. Here's how it works. Start with $y = ax^2 + bx + c$ and divide both sides by a. (Remember a is different from 0, otherwise we would have the linear equation $y = bx + c$.) We get

$$
\begin{aligned}
y &= a\left(x^2 + \frac{b}{a}x\right) + c \\
&= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2 \\
&= a\left(x + \frac{b}{2a}\right)^2 + \left(c - a\left(\frac{b}{2a}\right)^2\right)
\end{aligned}
$$

Notice that this is in the form $y = a(x-h)^2 + k$, with $h = -\frac{b}{2a}$ and $k = c - a(\frac{b}{2a})^2$. This means that our parabola has vertex at $(-\frac{b}{2a}, c - a(\frac{b}{2a})^2)$.

3.1 Example 1

Find the axis of symmetry and the vertex of the graph of the function $f(x) = x^2 - 6x + 11$. Solution: List the coefficients $a = 1$, $b = -6$, and $c = 11$. By completing the square for the equation $y = x^2 - 6x + 11$, we find

$$\begin{aligned} y &= (x^2 - 6x) + 11 \\ &= (x^2 - 6x + 9) + 11 - 9 \\ &= (x-3)^2 + 11 - 9 \\ &= (x-3)^2 + 2. \end{aligned}$$

Therefore, the parabola faces upward (a is positive) and it has a vertex at (3, 2) with a vertical line of symmetry $x = 3$.

3.2 Example 2

Find the axis of symmetry and the vertex of the graph of the function $f(x) = -2x^2 + 16x - 35$. Solution: List the coefficients $a = -2$, $b = 16$, and $c = -35$. By completing the square for the equation $y = -2x^2 + 16x - 35$, we find

$$\begin{aligned} y &= -2(x^2 - 8x) - 35 \\ &= -2(x^2 - 8x + 16) - 35 + 2 \cdot 16 \\ &= -2(x-4)^2 - 35 + 32 \\ &= -2(x-4)^2 - 3. \end{aligned}$$

Therefore, the parabola faces downward (a is negative) and it has a vertex at (4,-3) with a vertical line of symmetry $x = 4$. Notice that the parabola has a sharper steepness than the one in the previous example, because $a = 2$.