Exercises — Trigonometry

1. The Greeks "charted the heavens" using trigonometry as outlined below. Explain the details.
   (a) The ratio of the distance to the sun and the distance to the moon can be determined by measuring the angle between the sun and the moon at half moon.
   (b) The radius of the earth can be determined by measuring the angle to the horizon from the top of a mountain.
   (c) When the moon is straight above some point P on the equator, walk along the equator until the moon is no longer visible and call this point Q; the moon will be out of sight precisely when the angle between the centre of the earth C, the point Q, and the moon M is right; by measuring the difference in longitude between P and Q we know the angle QCM, and we already know the radius of the earth QC, so we can compute the distance to the moon EM.
   (d) Knowing EM, if we were standing at the moon we could measure the angle CMQ and compute the radius of the earth QC; so now we invert the roles of the earth and the moon and calculate the radius of the moon.
   (e) The same procedure of course applies to the sun.
   (f) Distance from Venus (or Mercury) to the sun: continually measure the angle VES; when it's at a maximum the angle EVS will be right, and we know ES so we can find VS.
   (g) Could the "size of the universe" have been reasonably estimated in early times if the earth had no moon?
2. Heron presented a method for digging a tunnel through a mountain from both ends simultaneously: to dig a north-south tunnel from a point N to a point S, find a point E east of the mountain such that NES is a right angle, then measure the lengths NE and SE, now calculate the angles ENS and ESN and give these values to the diggers. Explain how this works.
3. To apply trigonometry we must know the sine and cosine of many angles.
   (a) In principle this can be done by constructing appropriate triangles and measuring ratios of side lengths. How would you determine $\sin(10^\circ)$ for instance?
   (b) Determine $\sin(10^\circ)$ in this way and compare it with the actual value of $\sin(10^\circ)$. Comment on the accuracy of this method.
   (c) $\star$ Use the unit circle geometric definition of the sine and tangent functions to show that if $x > y$ then $\frac{\sin(x)}{x} < \frac{\sin(y)}{y}$ and $\frac{\tan(x)}{x} > \frac{\tan(y)}{y}$.
   (d) $\star$ Show that $\frac{1}{60} < \sin(1^\circ) < \frac{1}{45}$ by applying these inequalities to a 30°–60°–90° triangle.
   (e) Could you use the addition formulas to find bounds for $\sin(2^\circ)$, $\sin(3^\circ)$, $\sin(4^\circ)$ and so on from these bounds on $\sin(1^\circ)$?
   (f) $\star$ How do calculators know the values of the trigonometric functions?