Applications of Logarithms: Growth and Decay

EMLS 30

1 Objectives

To learn how logarithms are used to solve growth and decay problems. Related topics: 1, 3-5, 9, 10, 13, 25, 30.

- **Recall 1:** Consider the following problem:

  A population P of 1000 individuals is expected to grow exponentially over the next ten years at 4% per year. How large will the population be at the end of 10 years?

  This is the sort of problem that requires an understanding of exponentials. We solve it by noting that the population starts with $P_0 = 1000$ individuals and grows to

  $$P = 1000 + 0.04(1000) = 1.04(1000)$$

  by the end of the first year. To find the population at the end of the second year we add 4% of the population at the beginning of the second year to the population that was already there at the beginning of the second year. So we have

  $$P = 1.04(1000) + 0.04(1.04(1000)) = (1.04)^2(1000)$$

  individuals at the end of the second year. We see that at the end of the nth year the population will be

  $$P = (1.04)^n(1000).$$

  So at the end of ten years, $P = (1.04)^{10}(1000) = 1480$ individuals.

- **Note**

  If the starting population is $P_0 = 2000$ individuals, the population P at the end of n years will be $P = (1.04)^n(2000)$. If the starting population is P_0, the population at the end of n years will be $P = (1.04)^n P_0$.

  If the starting population is P_0 and the rate of growth is 5%, instead of 4%, the population at the end of n years will be $P = (1.05)^n P_0$.

  Likewise, if the rate of growth is r% and the starting population is P_0, then the population at the end of n years will be

  $$P \quad = \quad P_0(1 + \frac{r}{100})^n. \tag{1}$$

- **Recall 2:** A population P of 1000 individuals is expected to grow exponentially over the next ten years at 4% per year. How many years will it take for the population to reach 2000?

  To answer this question we look to Equation 1. The original population P_0 is 1000. The rate of growth $r\%$ is 4%. The number of years n is our unknown. So

  $$2000 = 1000(1.04)^n.$$

  Dividing both sides of this equation by 1000 gives $(1.04)^n = 2$.

  Now, recall that if $a^x = y$, then $x = \log_a y$. Using this to solve for n in the above equation gives $n = \log_{1.04} 2$.

  How will you calculate $\log_{1.04} 2$ if you have a calculator that only calculates $\ln x$ or $\log x$? Recall that

  $$\log_b c = \frac{\log_a c}{log_a b}.$$

  Letting $a = e$, $b = 1.04$ and $c = 2$ gives

  $$\log_{1.04} 2 = \frac{\ln 2}{\ln 1.04} = 17.67,$$

  so the number of years it takes for the population to reach 2000 is about 17.67 years.

- **Introduction 1** (*Decay*):

  The questions that were answered above are classified as growth questions. For example, "how long will it take for a population that increases at 4% per year grow to x individuals?" is a growth question. Note that the word population can be replaced by any other word that indicates some collection of things; i.e., money, lilies in a pond, raindrops, etc.

  The answer to the question in Recall 1 was $P = 1000(1.04)^{10} = 1480$. But if the question were about a population that shrinks at a rate of 4% per year then the answer would have been $P = 1000(1 - .04)^{10} = 665$. Note that *a negative exponent models shrinking or decaying.*

  Consider the following question: A certain radioactive substance has a half-life of 40 minutes. (That is, the time it takes for the substance to reduce to half of the original amount is 40 minutes.) What fraction of an initial amount of this substance will remain after 1 hour and 20 minutes?

  The model for decay is

  $$P(n) = P_0 e^{kn},$$

  where $P(n)$ represents the amount of substance (or population of atoms) after n years, P_0 represents the initial amount, and k is some constant that depends on what substance we are talking about. If k is negative, the population will decay; if it is positive, it will grow. The trick is first to find k.

We are given that half of the substance will remain when $n = 40$ minutes. Therefore,

$$P(40) = \frac{1}{2}P_0 e^{40k},$$

so $40k = \ln(\frac{1}{2})$ and $k = \frac{-\ln 2}{40}$. This means that we know the population at n years to be

$$P(n) = P_0 e^{\frac{-ln2}{40}n}.$$

Notice that we could also write this last function as

$$P(n) = P_0 2^{-n/40},$$

or

$$P(n) = P_0(\frac{1}{2})^{n/40}.$$

If we change the half-life from 40 to 50 minutes, we find the population at n years to be $P(n) = P_0 2^{-n/50}$. And if the half-life is h minutes, the population at n years will be

$$P(n) = P_0 2^{-n/h}.$$

We may now answer the original question, which asks for $P(80)$, the amount of the substance left after 80 minutes. From our model, we know that

$$\frac{P(80)}{P_0} = 2^{-80/40} = 2^{-2} = \frac{1}{4}.$$

- **Keep the Form in Mind**

1. The form of the function that gives the population is

$$P(n) = P_0 2^{-n/h},$$

where h is the half life. The 2 is conspicuously suggestive of half-life or doubling time. If a population doubles every d years, then the population at the end of n years is

$$P(n) = P_0 2^{n/d},$$

where d represents the number of years it takes for the population to double from its original size.

2. If h represents the third-life (the time it takes for the population to shrink to one third its size) then the population at the end of n years is given by

$$P(n) = P_0 3^{-n/h}.$$

3. You may see population functions displayed as $P(n) = P_0(\frac{1}{2})^{n/h}$. This is just another way of expressing $P(n) = P_0 2^{-n/h}$ and therefore just another way of expressing population using the half-life h.

2 Examples

1. You put $\$1,000$ in a money market fund. It takes 10 years for your money to double. Approximately how long will it take for your money to triple?

   Solution: The population (amount) of money is given by $P(n) = P_0 2^{n/10}$. (See Introduction 1)

   Let n be such that $P(n) = 3P_0$. Then $3 = 2^{n/10}$. Therefore,

   $$\log_2 3 = \frac{n}{10},$$

   so

   $$n = 10 \log_2 3 = 15.85.$$

   So it will take about 16 years for your money to triple.

2. A certain radioactive substance has a half-life of 40 minutes. What percentage of an initial amount of this substance will remain after 2 hours and 40 minutes?

   Solution: Again, $P(n) = P_0 2^{-n/40}$. The question asks us to find $\frac{P(160)}{P_0}$, which may be found from the equation

   $$\frac{P(160)}{P_0} = 2^{-160/40} = 2^{-4} = \frac{1}{16} = 6.25\%.$$

3. You put $\$1,000$ in a money market fund at r percent interest compounded annually. After 5 years you have $\$2,000$ in your account. What is the interest rate r?

   Solution: You know that $\$2000 = 1000(1 + \frac{r}{100})^5$ and hence the equation we want to solve is $2 = (1 + \frac{r}{100})^5$. The equation

   $$2^{\frac{1}{5}} = 1 + \frac{r}{100}$$

   becomes $100(2^{\frac{1}{5}} - 1) = r$, so $r = 14.87$ percent.

4. A certain radioactive element has a half-life of 900 years. Starting with 30 milligrams there will be $q(t)$ milligrams left after t years, where $q(t) = 30(\frac{1}{2})^{kt}$. How much will be left after 2500 years?

   Solution: We know that $q(t) = 30(\frac{1}{2})^{kt}$. Therefore, $15 = 30(\frac{1}{2})^{900k}$. Solving for k gives

   $$\frac{1}{2} = 30(\frac{1}{2})^{900k},$$

   so $k = \frac{1}{900}$. Hence, $q(2500) = 30(\frac{1}{2})^{2500 \cdot \frac{1}{900}} = 30(\frac{1}{2})^{25/9} = 4.37$ milligrams.

5. Charcoal remains, found in an archeological site, were found to contain only 65% of the carbon-14 in a living snail. How old is the site? The half-life of carbon-14 is 5668 years.

   Solution: We know that $P(n) = 0.65 = 2^{-n/5668}$. Hence, $\log_2 0.65 = \frac{-n}{5668}$. Solving for n gives

$$n = -(5668) \log_2 0.65 = 3522.6.$$

   Therefore, the site is at least $3,522.6$ years old.

3 Exercises

1. A population of 1000 is expected to grow exponentially over the next ten years at 4% per year. How large will the population be at the end of 10 years?

2. A certain radioactive substance has a half-life of 30 seconds. What percentage of an initial amount of this substance will remain after 10 seconds?

3. If you invest $\$5,000$ in a money market fund at 15 percent interest compounded annually, how much will you have after 10 years?

4. Approximately how long will it take for money to accumulate to twice its value if it is invested at 8%, compounded annually?

5. Suppose that you invest P dollars at $r\%$ compounded annually. Write an expression for the amount accumulated after n years.

6. A certain radioactive element has a half-life of 400 years. Starting with 50 milligrams there will be $q(t)$ milligrams left after t years, where $q(t) = 50(\frac{1}{2})^{kt}$. How much will be left after 2257 years?

7. Bones from a human body were found to contain only 76% of the carbon-14 in living bones. How long before did the person die? The half-life of carbon-14 is 5668 years.

8. Suppose that you received a gift of $\$10,000$ this year and placed it in an fund that accrues interest at 10% compounded annually. If you do not withdraw it until the year 2040, how much will you have then?

4 Solutions

1. 1480.24 individuals

2. 79%

3. $\$20,227.80$

4. 9 years

5. $P(1 + \frac{r}{100})^n$
6. 1 milligram
7. $2,244$ years
8. $\$452,593$ or less, depending on what year it is right now.