Systems of Linear Equations I

EMLS 31

A *system* of two linear equations in two variables is formed by two equations:

$$Ax + By = C$$

$$Ex + Fy = G.$$

There are two main techniques for solving these.

I. Substitution:

Consider the following system:

$$x + y = 2$$

$$2x - y = 7$$

1. Solve one of the equations for one variable in terms of the other:

$$y = -x + 2$$

2. Substitute the expression above into the other equation:

$$\begin{aligned} 2x - y &= 7 \\ 2x - (-x + 2) &= 7 \\ 2x + x - 2 &= 7 \\ 3x - 2 &= 7 \\ 3x &= 9 \\ x &= 3. \end{aligned}$$

3. Substitute the solution above into one of the equations in the system:

$$\begin{aligned} x + y &= 2 \\ 3 + y &= 2 \\ y &= 2 - 3 \\ y &= -1. \end{aligned}$$

4. Check to see if the solutions obtained are right.

$$\begin{aligned} x+y &= 2 \\ 3-1 &= 2 \\ 2x-y &= 7 \\ 2(3)-(-1) &= 7 \\ 6+1 &= 7 \end{aligned}$$

II. Elimination by Addition:

Consider the system of equations

$$x+y=2$$

$$2x-y=7.$$

1. Multiply one of the equation with an appropriate real number that will eliminate one of the variables when the equations are added together.

   Here, we multiply the first equation by 1 and add the first equation to the second.

$$\begin{array}{rcl} \text{x + y} & = & 2 \\ \text{+2x - y} & = & 7 \\ \hline \text{3x} & = & 9 \end{array}$$

   So $x=3$ is our x value.

2. Substitute $x=3$ into one of the equations to find the value of y.

$$\begin{aligned} x &= 3 \\ x+y &= 2 \\ 3+y &= 2 \\ y &= -1 \end{aligned}$$

3. Check that the solutions work.

$$\begin{aligned} x+y &= 2 \\ 3-1 &= 2 \\ 2x-y &= 7 \\ 2(3)-(-1) &= 7 \\ 6+1 &= 7. \end{aligned}$$

System of Three Equations in Three Variables:

$$\begin{aligned} Ax+By+Cz &= D \\ Ex+Fy+Gz &= H \\ Ix+Jy+Kz &= L \end{aligned}$$

Solving systems of three equations in three variables is similar to solving systems with two variables. The solution set is an ordered triple of real numbers (x, y, z).

Example:

Consider the system of equations:

$$\begin{aligned} 2x + 3y + z &= 1 \\ 5x + 2y - 3z &= 8 \\ x - 4y - z &= 18. \end{aligned}$$

1. Multiply the first equation by 3 and add the result to the second equation:

$$\begin{array}{rcl} 3(2\text{x} + 3\text{y} + \text{z}) & = & 3 \\ +5\text{x} + 2\text{y} - 3\text{z} & = & 8 \\ \hline 11\text{x} + 11\text{y} & = & 11. \end{array}$$

   Factoring out 11 from the solution gives $x + y = 1$.

2. Add the first and third equations:

$$\begin{array}{rcl} 2\text{x} + 3\text{y} + \text{z} & = & 1 \\ + \text{x} - 4\text{y} - \text{z} & = & 18 \\ \hline 3\text{x} - \text{y} & = & 19. \end{array}$$

3. Now, add the solutions from parts 1 and 2:

$$\begin{array}{rcl} \text{x} + \text{y} & = & 1 \\ +3\text{x} - \text{y} & = & 19 \\ \hline 4\text{x} & = & 20 \\ \text{x} & = & 5. \end{array}$$

4. Substitute $x = 5$ into one of the two equations from parts 1 and 2 to obtain y:

$$\begin{aligned} x + y &= 1 \\ 5 + y &= 1 \\ y &= -4. \end{aligned}$$

5. Substitute $y = -4$ and $x = 5$ into one of the original three equations to get z:

$$\begin{aligned} x - 4y - z &= 18 \\ 5 - 4(-4) - z &= 18 \\ 5 + 16 - z &= 18 \\ 5 + 16 - 18 &= z \\ 3 &= z. \end{aligned}$$

   The solution is $(5, -4, 3)$.

1 Exercises

Solve the following systems by substitution:

1. $\begin{aligned} x - 2y &= 0 \\ 3x + 2y &= 8 \end{aligned}$

2. $\begin{aligned} x + 6y &= 19 \\ x - 7y &= -7 \end{aligned}$

3. $\begin{aligned} 8x + 5y &= 100 \\ 9x - 10y &= 50 \end{aligned}$

4. $\begin{aligned} 2x + 5y &= 29 \\ 5x + 2y &= 13 \end{aligned}$

5. A combined total of \$12000 is invested in two bonds that pay 8.5% and 10% simple interest. The total annual interest is \$1140. How much is invested in each bond?

6. Find the 2 integers satisfying the following: the sum of the larger number and twice the smaller number is 61 and the difference of the two numbers is 7.

Solve the following systems by elimination:

7. $\begin{aligned} x - 3y &= 7 \\ -2x + 6y &= -14 \end{aligned}$

8. $\begin{aligned} 4x + 3y &= 8 \\ x - 2y &= 13 \end{aligned}$

9. $\begin{aligned} 3x - 2y &= -20 \\ 5x + 6y &= 32 \end{aligned}$

10. $\begin{aligned} -2x + 5y &= 5 \\ 4x - 10y &= 9 \end{aligned}$

11. $\begin{aligned} x + y + z &= -3 \\ 4x + y - 3z &= 11 \\ 2x - 3y + 2z &= 9 \end{aligned}$

12. $\begin{aligned} x + y - z &= 2 \\ x + 2y - 4z &= 3 \\ -2x - 2y + 2z &= 5 \end{aligned}$

13. $\begin{aligned} x + 6y + 2z &= 9 \\ 3x - 2y + 3z &= -1 \\ 5x - 5y + 2z &= 7 \end{aligned}$