PROOF WRITE-UP

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Theorem: If $f(x) = g(x) - j(x)$, for "well behaved" functions g and h, then $f'(x) = g'(x) - j'(x)$?

Well, first of all, why would we want to prove this? This is a rule in calculus which is sometimes referred to as the 'difference rule'. In actuality, it's a subset of the sum rule. The point of all of these rules is to be able to take derivatives efficiently without having to worry about the technical definition of a derivative each time. Before we can do this, however, we need to make sure that the rule works and understand why it works the way it does.

In order to prove the above theorem, we need to return to our definition of a derivative. Remember, the derivative of a function $f(x)$ can be described as: $\underset{h\to 0}{Limit} \frac{f(x+h)-f(x)}{h}$. Now, let's define our function $f(x)$ as equal to $g(x) - j(x)$ So, in order to prove that $f'(x) = g'(x) - j'(x)$ we need to prove that $f'(x) = \underset{h\to 0}{Limit} \frac{g(x+h)-g(x)}{h} - \underset{h\to 0}{Limit} \frac{j(x+h)-j(x)}{h}$. Specifically, we need to prove that, by definition, the derivative of our function f is equal to the difference between the derivatives of g and j.

Using our definition of a function, we can state that:

$$\begin{aligned} f'(x) &= \frac{Limit}{h \to 0} \frac{g(x+h) - j(x+h) - (g(x) - j(x))}{h} \\ &= \frac{Limit}{h \to 0} \frac{g(x+h) - g(x) - j(x+h) + j(x)}{h} \\ &= \frac{Limit}{h \to 0} \left(\frac{g(x+h) - g(x)}{h} - \frac{j(x+h) - j(x)}{h}\right) \end{aligned}$$

Because the function is well behaved, we can distribute the limit, leaving us with:

$$f'(x) = \frac{Limit}{h \to 0} \frac{g(x+h) - g(x)}{h} - \frac{Limit}{h \to 0} \frac{j(x+h) - j(x)}{h} = g'(x) - j'(x)$$

So, $f'(x) = g'(x) - j'(x)$ as predicted. Now that we've proven out theorem, we can use it to quickly find derivatives involving subtraction. For example, the derivative of $x^3 - x^2$ is $3x^2 - 2x$. Using this rule, see if you can find the derivatives of the following functions:

1. $f(x) = x^4 - 3x^3$
2. $f(x) = 5x^4 - 3x^2 + 4$
3. $g(y) = 6y - 4$
4. $f(x) = 2(3x^3 + 4x^2 - 4)$
5. $f(x) = x(6x^4 + 2x^2 - x)$

References

[1] W. M. Priestly. *Calculus: A Liberal Art.* Springer, New York, 2nd edition, 1998.