Block Labelings and the Oberwolfach Problem

Devin Willmott

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1 Introduction

Say a group of seven people wish to arrange themselves around a pair of tables - one that seats three people, the other seating four. Additionally, they wish to arrange themselves multiple times such that, at the end of the night, each person has sat next to each other person exactly once. We can see below that, if we assign each person a number, there exists a set of configurations that fulfill these requirements:

[1 4 7] [2 3 5 6]
[2 5 7] [3 4 6 1]
[3 6 7] [4 5 1 2]

The ability to come up with a set of seating arrangements for a number of tables with the property that each pair of guests sits next to one another exactly once is known as a solution to the particular *Oberwolfach problem*. For a number of tables t, their sizes are denoted as $k_1, k_2, \ldots, k_t$. From the sizes of the tables we can glean both the total number of participants in each arrangement ($n = k_1 + \ldots + k_t$) and the required number of distinct arrangements (each person sits next to two different people in each new arrangement, so the seating must be done $\frac{n-1}{2}$ times). Thus, all of the relevant information is included in the notation for the Oberwolfach problem, $OP(k_1, \ldots, k_t)$. The above example exhibits a solution to OP(3,4).

M. A. Ollis and D. A. Preece showed in [1] that there exists a solution to $OP(r, r, s)$ for odd r and sufficiently large s. In this paper, we will adapt their proof to show that a solution exists for all $r \geq 4$. To do so, we must first become acquainted with a few more combinatorial objects.

2 Definitions

We define $a = (a_1, a_2, \ldots, a_n)$ as a sequence consisting of the elements $0, 1, \ldots, n-1$ of $\mathbb{Z}_n$. Additionally, we can describe $b = (b_1, b_2, \ldots, b_{n-1})$ with the definition $b_i = a_1 - a_{i+1}$. If b contains two occurances from $\{a, a^{-1}\}$ for each $a \in \mathbb{Z}_n$, then a is a *terrace* and b is its *2-sequencing.* If b simply contains each element from 1 to $n-1$, the sequence a is a *directed terrace.* Terraces exist for many groups

Example 1 *An example terrace is shown below, with its 2-sequencing under it.*

3 6 5 7 2 4 0 1
3 7 2 3 2 4 1

The *translate* t of a terrace a is the sequence resulting when we add an element $x \in \mathbb{Z}_n$ to each element in a. Because x is added to each element b_n, the sequencing b is unaffected by translating, and thus t is also a terrace. If the first element of a terrace is 0, the terrace is *basic.*

Example 2 *The basic translate of the terrace in Example 1 is*

0 3 2 4 7 1 5 6 .

We can also consider a as a sequence in $\mathbb{Z}$; changing the definition of b to $b_i = |a_{i+1} - a_i|$, we will describe a as a *graceful labeling* and b as its difference list if, once again, b contains all of the elements from 1 to $n-1$.

Example 3 *The sequence below is both a directed terrace and a graceful labeling; below it is its difference list in $\mathbb{Z}$:*

5 0 7 1 4 6 2 3
5 7 6 3 2 4 1

Example 4 *Finally, the following sequence is an example of a terrace that is not a graceful labeling; the sequence is presented first with its 2-sequencing and then with its difference list.*

3 7 0 6 5 2 4 1
4 1 6 7 5 2 5

3 7 0 6 5 2 4 1
4 7 6 1 3 2 3

3 Match Points

The essential condition for translating graceful labelings into solutions to the Oberwolfach problem is the existence of a *match point*. The notion of a match point is native to terraces; when $a = (a_1, a_2, \ldots, a_n)$ is a terrace and $b = (b_1, b_2, \ldots, b_n)$ is the 2-sequencing of a, we call i as a match point if $a_i = b_i$ [2].

Example 5 *The terrace shown in Example 1 has a match point at* $i = 5$*, as* $a_5 = b_5 = 3$*. It also has match points at* $i = 1$ *and* $i = 6$*.*

The following theorem details how to construct graceful labelings that can be translated into terraces with match points.

Theorem 1 *[1] Let* $a = (a_1, a_2, \ldots, a_n)$ *be a graceful labeling with the property* $a_1 - a_r = a_r - a_{r+1}$*. Then* $\mathbb{Z}_n$ *has a basic terrace with a match point at* r*.*

Proof. Consider a as a terrace; adding the element $x = -a_1$ to a makes the resulting terrace basic. The value of $a_r - a_1$ is therefore simply a_r, and so the difference b_r is $a_r - a_{r+1} = a_1 - a_r = a_r$, and a has a match point at r. □

We now present a construction for graceful labelings that yield terraces with even match points.

Theorem 2 *There exists a terrace of order* $n = 2 \bmod 3$ *with a match point at* $r = 2(n+1)/3$*.*

Proof. The graceful labeling g of order $n = 2 \bmod 6$ that follows the construction

$$a_i = \begin{cases} \frac{n+1-3i}{2} & \text{for odd } i \leq \frac{n+1}{3} \\ \frac{n-2+3i}{2} & \text{for even } i \leq \frac{n+1}{3} \\ \frac{3(i-1)-n}{2} & \text{for odd } i > \frac{n+1}{3} \\ \frac{3(n-i)+2}{2} & \text{for even } i > \frac{n+1}{3} \end{cases}$$

contains a match point. By the construction, $a_1 = \frac{n+1-3(1)}{2} = \frac{n}{2} - 1$, $a_{\frac{2(n+1)}{3}} = \frac{3n-2(n+1)+2}{3} = \frac{n}{2}$, and $a_{\frac{2(n+1)}{3}+1} = \frac{2(n+1)+3-3-n}{2} = \frac{n}{2} + 1$. Clearly, $|g_1 - g_r| = |g_r - g_{r+1}| = 1$ and so, by Theorem 1, there exists a basic terrace of order n with a match point at $2(n+1)/3$.

When $n = 5 \bmod 6$, the property $|g_1 - g_r = g_r - g_{r+1}|$ holds when parity is reversed in the above construction. It is easily verified that $a_1 = \frac{n}{2} + 1$, $a_{\frac{2(n+1)}{3}} = \frac{n}{2}$, and $a_{\frac{2(n+1)}{3}+1} = \frac{n}{2} - 1$, and a basic terrace with match point is guaranteed through the same theorem.