Rubik's Cube
and
a philosophy of physics

by

James Henri Mahoney

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ABSTRACT

How a physical system may be understood is discussed informally by analyzing a puzzle called "Rubik's Cube." This understanding, when formalized, may take the form of a scientific theory in which invariants and mathematical models describe the properties of the system.

Questions of viewpoint and elegance are seen to have primary importance in the creation of these theories. Any mode of thought implicitly makes assumptions and focuses on only some parts of the problem; therefore, a full understanding requires the ability to sift perspectives. The most useful and beautiful way of thinking about the Cube turns out to be the most elegant.

The specific aspects of the NxNxN Cube studied are the number of possible positions, the relation between positions and move sequences, and matrix representations of the Cube.

Thesis Supervisor: Dr. Daniel J. Kleitman

Title: Professor of Applied Mathematics

Dedicated to

God,

"for God is the simplest of all."

Leonard Bernstein`s "Mass"

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Preface

One fateful day during the winter of 1979, I stopped in to a games store and bought a cute-looking little puzzle. This may have been a mistake, since I then spent days- no, months- poring over the thing. About a year later, when I thought I had essentially figured out how it worked, I was asked to pick a topic for my undergraduate physics thesis. Since I had already spent more time playing with the Cube than on any two courses, essentially creating my own self-contained scientific theory (complete with hunches, wrong guesses, fundamental laws and all the rest), this "puzzle" seemed to be the perfect choice.

The tricky part was finding a thesis advisor. After some legwork and about fifteen explanations of my idea, I finally found someone who answered " Sure. Sounds like fun."

Hence this paper.

I. Definitions

- Physics

What is "physics?" One dictionary said it was "a science that deals with matter and energy and their interactions in the fields of mechanics, acoustics, ...", where "science" was defined as "a knowledge covering general truths of general laws, especially as obtained and tested through scientific method." Others might say that physics is a collection of models describing the universe. Most undergraduates probably see physics as a problem solving technique.

None of these explanations are wrong, but there is a simpler one. Physics may be thought of as the study of systems, the attempt to find the vantage points from which the world can be seen. Two points are important. One: the object of the game of physics is "understanding." Problem/solution modes of thought may be ways of attaining this goal, but are not the end in themselves. Physicists are not engineers. Two: the key to this understanding lies in the viewpoint which is chosen. For example, looking at a collection of 100,000,000 interacting particles as a complicated n-body problem is not very helpful, while treating the ensemble probabilistically leads to interesting results.

Now the question becomes one of procedure. How is this done? Given a system, which contortion of it will make it appear simple and intuitive? Well, it depends on the system. However, there exist some very general and powerful techniques which are often used. The ones which will be discussed here are modeling and invariants.

The most basic way of understanding any system is to find some kind of analogue. A similar system, one which is already familiar and which shares some essential characteristics with the original system, should behave in a similar fashion. This technique of comparing an unknown with a known is very common. For example, in quantum theories, electrons and photons are sometimes thought of as waves, and sometimes as particles. Neither is entirely correct, but some description using familiar concepts is necessary to make any progress at all. Mathematical systems are very precisely defined, and understood (usually) quite well; therefore, they are a powerful tool for use in this type of modeling. Not without reason is mathematics said to be the "language of physics."

Another way of learning about the properties of a physical system is to find and study its invariants. Actually, the invariants have to be invented, i.e. variables must be defined which do not change under transformations characteristic to the system in question. Strangely enough, these variables tend to become incredibly important in understanding what is happening to the system. Moreover, they start to take on a life of their own. After a while, the invariants seem to be almost "real," as if they had always been the natural parameters of the system- in spite of the fact that they are simply defined quantities with certain properties.

Classical mechanics provides a good example. The "total kinetic energy" and the "total momentum" of a group of isolated particles are invariants of that system. Even though they are only two of the thousands of variables which can be defined on the group of particles, these somehow describe the particles in a significant way. Why? Because the energy and momentum of a system of isolated particles are invariants of that system; they do not change with time. Finding analogous invariants in other systems can be crucial to an understanding of the systems.

After some analogues and invariants are found, the viewpoint created seems obvious, almost inevitable. Unfortunately, this is not the case. The chain of reasoning and guesswork leading to a given theory is never easy to invent, only easy to follow. Nor is any particular theory the only description of a given system. Max Plank started a train of thought with the then unexplained black body spectrum, and ended with quantized energy levels. Today, looking back, his logic is clear; however, at the time, the things he did seemed almost magic. He was able to look at the problem in a new way.

This paper will show how a particular structure, a physical/mathematical puzzle called "Rubik's Cube," may be "understood" by an application of the appropriate invariants and models. While simple enough to be at least partially treated in a paper of this scope, it is nonetheless complicated enough to allow for a variety of approaches. The object is to find the particular point of view from which the Cube becomes intuitive, or at least easily comprehended. Perhaps then a method of conducting such searches in general will become a little clearer.

- the Cube

Ideally, at this point the reader would be handed a Cube and told "Here. This is a Cube." Anything more already starts advocating a particular point of view. However, since a Cube is unlikely to pop out of this paper, a description must be given. Right now the perspective will be as impartial as possible.

A "Rubik`s Cube," henceforth called "the Cube," is a physical object in the shape of a cube which can be manipulated so as to assume different positions. In the initial position, called "start" or "solved," each face of the cube is a different solid color. Although the Cube cannot be easily dismantled, it appears to be made of a 3x3x3 array of smaller cubes, called "cubies" (see fig.1a). Any plane of nine cubies may be rotated about the center cubie, generating the possible positions of the Cube. Fig. 1b shows a typical move.

fig. 1a)

the 3x3x3

Cube

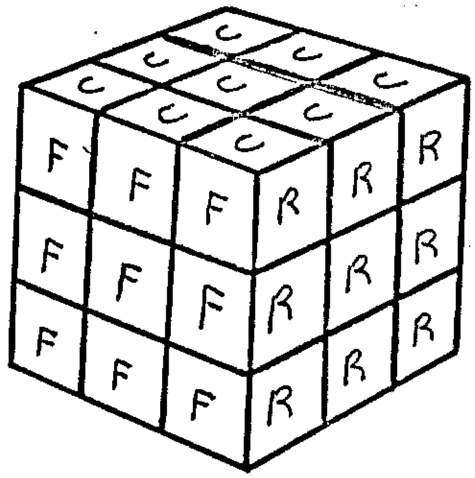
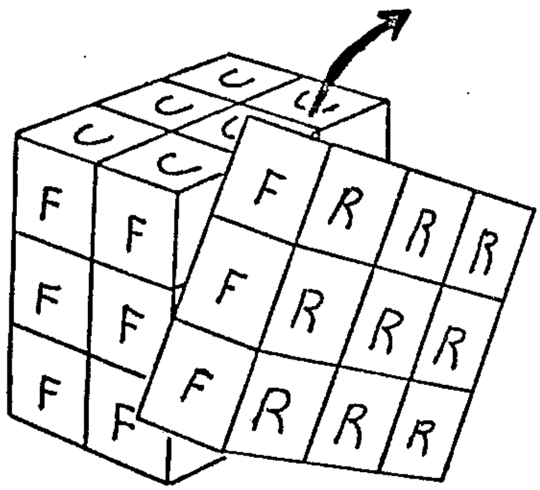


fig. 1b)

Halfway through an "R" move.



Now, what is to be "understood?" That is most of the problem: defining the problem. At a minimum, an understanding of the Cube should include the answers to questions like "what sequence of moves will take the Cube from position A to position B," or "how many possible positions are there?" Notice that already a point of view is being decided upon.

In order to talk about the transformations and pieces of the Cube, some descriptive notation must be found.

David Singmaster`s (see the acknowledgements) seems very natural and is used almost universally; therefore, it will be used first. This notation uses the letters U,D,F,B,R,and L (which are initials for Up, Down, Front, Back, Right, and Left) to mark the sides of the Cube, instead of colors (see fig. 1) . The cubies are labeled by the colors showing, using small letters; thus, a typical corner cubie is ufr. One quarter clockwise rotation, of a Cube side is denoted by the capitalized letter of the center square on that side. For example, the move shown in fig 1b is "R". A counter-clockwise rotation of the same side is called "R'".

This notation has assumed quite a bit. No mention has been made of a central, hidden cubie. If one exists, it is deemed unimportant. The orientation of the entire Cube in three space and of the center-face cubies has also been ignored. Most people who play with the Cube are primarily interested in "solving" it, i.e. restoring the original pattern of solid faces after the Cube has been randomized by some unknown series of moves. To solve this problem, nothing about how the Cube is sitting in space or how the face-centers have been turned matters, so Singmaster`s notation is a natural choice.

Cubists who think in this way picture the Cube something like this: each center-face , although it can rotate, is fixed in space. The only things which move around on the Cube are edges and corners. Each edge has twelve places where it can be and two orientations at each place, while each corner has eight places it can be with three orientations per place (see fig. 2). The plane slices through the center of the Cube don't rotate, instead, two

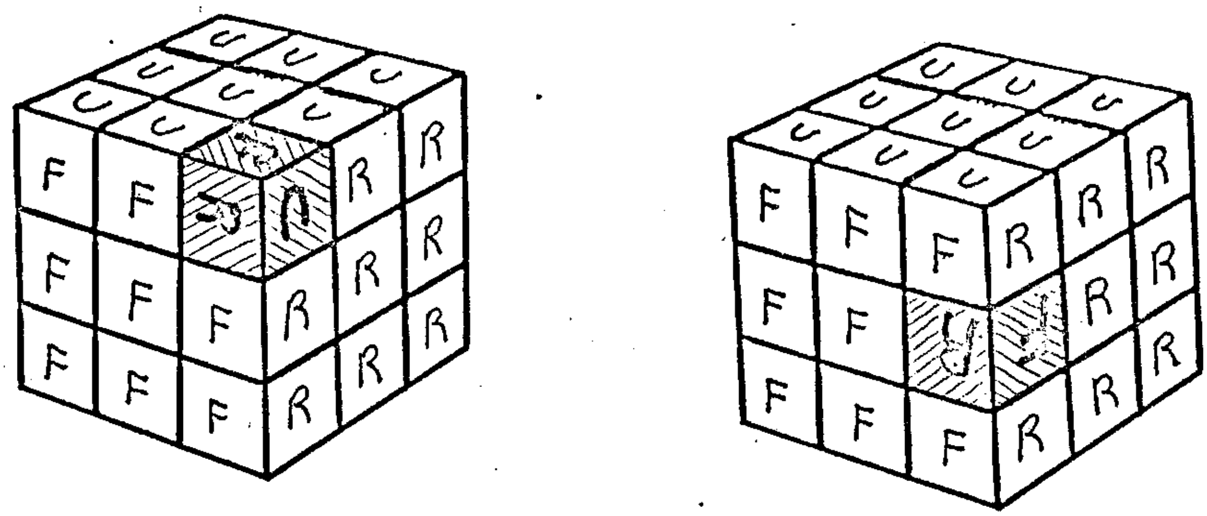


fig. 2) The ufr corner is turned clockwise and the fr edge is flipped. Both are still in their original locations.

opposite faces can both be rotated in the same direction.

The natural extension of this thinking is to consider the Cube as a finite group (large, but finite) generated by the operators U,D,F,B,R, and L. As was stated before, mathematical analogues are powerful ways to understand systems, especially systems which are as simple and elegant as the Cube. If all the properties of finite groups were completely understood then understanding the Cube would just be understanding a specific finite group. However, this paper does not have to become just an exercise in group theory, for there are other profitable ways of looking at the Cube.

The Cube might be thought of as being more random if the sides were mixed colors. If this could be quantized, perhaps the ideas of "entropy" and "temperature" might be applied to describe what the Cube would look like (on the average) after a number of arbitrary moves. Or perhaps the space of all positions could be treated as a network, so that the minimum moves from position A to position B would just be the shortest path connecting the two. Maybe even some kind of quantum energy levels could be visualized...

The point is that different viewpoints lead to different ways of understanding. Each point of view will make the answers to some questions intuitive, but other questions will appear to be difficult- even though in some other model, the first questions might be the harder ones . A group theory approach to the Cube makes "solving" the Cube a reasonable problem, but finding the possible positions is hardly easy in this language.

So, what point of view should be adopted? The one which makes whatever aspect of the Cube you are studying obvious. To proceed further, some part of the Cube universe must be singled out for understanding. Therefore, in the next section, a specific question will be asked, and an appropriate perspective will be proposed.

II. Positions

How many different positions can the Cube attain? Why? What about a 4x4x4 Cube? An NxNxN ? This section will answer these kinds of questions by finding some of the invariants of the Cube. The notation introduced in the last section and the implied viewpoint which focuses on the colors that can be seen on the exposed sides of the Cube (from now on referred to as the "S-view," for "side view") will be the starting language for the discussion, since this way of looking at the cube seems the most natural.

- the 3x3x3 (S-view)

As stated previously, the S-view keeps each center-face cubie in its own cubicle (a spot in space where a cubie can be), and moves the eight corners and twelve edges with twelve transformations : R,R',L,L', etc. Notice that the cubies have been divided into three types: edges, corners, and center-faces. Looking at the Cube and the moves, one sees easily that corners remain corners, and edges are always edges. This fact, that a cubie's type never changes, can be thought of as the first law of the Cube.

If this law were the only constraint on the positions of the edges and corners (the S-view ignores the face-centers since the stay in the same cubicle and always show the same color), then the total number of possible positions would be

(1) $(8!\ 3^8)(12!\ 2^{12})$ = first guess, possible positions of the 3x3x3 S-view Cube

or eight factorial (8x7x6...) permutations of the corners, times three orientations per corner raised to the eighth power since there are eight corners, times twelve factorial permutations of the edges, times two orientations per edge to the twelfth power (see any combinatorics text for details on counting if that sounded unreasonable).

However, many of these positions cannot be reached by any sequence of the twelve basic moves. For example, the two corners lrt and lrb cannot be exchanged (leaving ther rest of the Cube untouched) without dismantling the Cube. Why not? Because flippiness, turniness, and parity are conserved.

"Parity" is a variable which describes whether the net permutation of the edges and corners is even or odd. A permutation of a group of objects is said to be "even" if it takes an even number of 2-element exchanges to get to it, and "odd" if it takes an odd number of exchanges. For example, [2 1 3] is an odd permutation of [1 2 3] because one switch is required; 1 and 2 swap places. The parity of the Cube is defined to be 0 if the edge-corner permutations are even-even or odd-odd, and 1 otherwise.

At first glance, this variable may seem worthless and artificial; however, it has one important property which it shares with the other invariants: legal transformations of the Cube do not change its value. To see this, just examine how a typical move permutes the edges and corners. Using the numbers 1-8 to represent the corners, the starting position of the Cube can be thought of as [1 2 3 4 5 6 7 8]. After one move, four digits have cycled; therefore the position is something like [2 3 4 1 5 6 7 8]. Getting this same string by just swapping pairs of numbers takes three (or some odd number of) exchanges : [1 2 3 4] to [2 1 4 3] to [2 4 1 3] to [2 3 4 1]. This means that [2 3 4 1 5 6 7 8] is an odd permutation of [1 2 3 4 5 6 7 8]. The same reasoning works on the edges; one move again cycles four cubies.

The parity of the Cube`s starting position is 0, since both the corners and edges are in an even permutation (initial position). After any one move, the corner and edge permutations are odd, so the parity is again 0. After n moves, the permutations are still either both odd (if n is odd); or both even (if n is even); therefore, all Cube positions with parity 1 - half of the positions counted in equation (1) - cannot be reached by legal moves. In particular, two corners cannot be swapped without changing the edges.

"Turniness" describes the orientations of the eight corners, regardless of their locations. Every corner

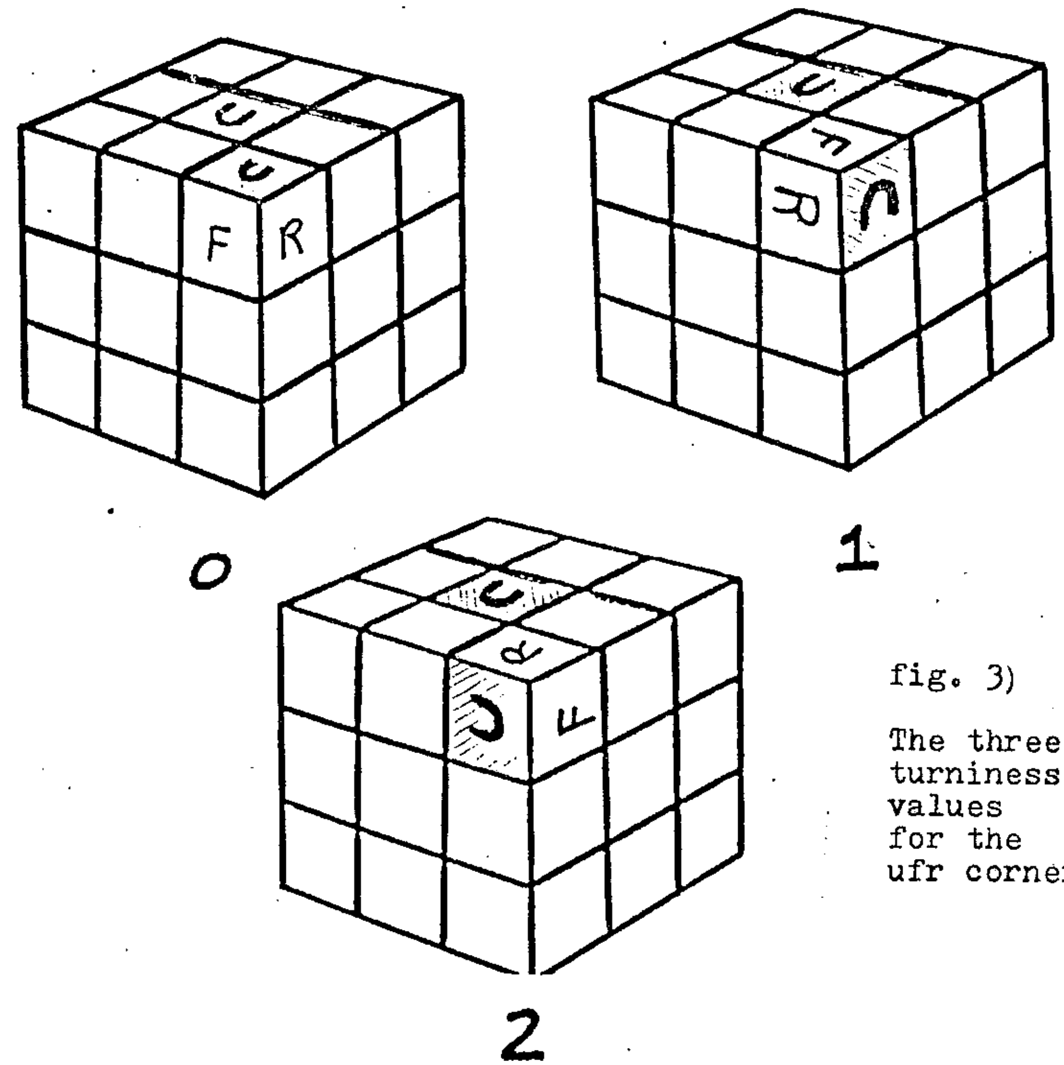


fig. 3)

The three turniness values for the ufr corner.

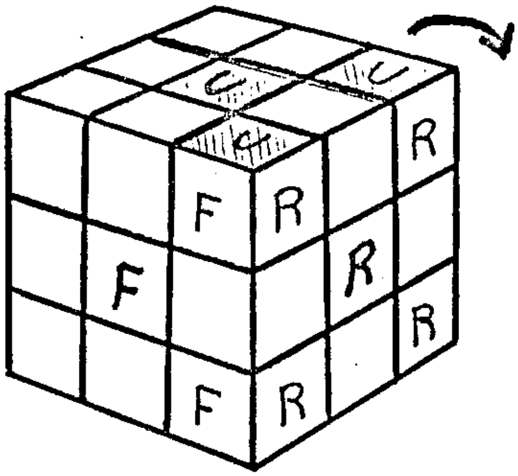
starts out on the top or bottom of the Cube, so every one has a up side or a down side. The orientation of any corner is defined to be 0 when the up side (or down side) of the corner is parallel to the U face center, 1 when the up (or down) side appears to be rotated clockwise one-third of a turn away from the 0 position, and 2 when rotated two-thirds of a turn. Fig. 3 makes this clear. The turniness of the Cube is the sum of all eight corner orientation values, mod 3. Note that the solved position of the Cube has turniness

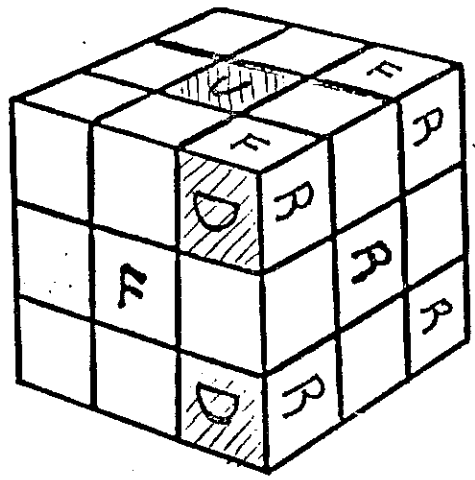
0.

Again the question "how does one move affect this quantity?" is asked. The U and D moves do not change the relative positions of the u and d center faces and the corners; therefore, they leave the turniness (and the corner orientation values) unchanged. Any of the other four moves affect the corners in the same way, illustrated in fig. 4. As shown, one of the corners on the top of the Cube moves to the bottom, and the other one on top stays on top. This last corner changes from orientation 0, 1, or 2 to orientation 1, 2, or 0; 1 is added to the corner's value (everything is done mod 3 here). The corner that moves to the bottom of the Cube shifts from 0, 1, or 2 to 2, 0, or 1; 2 is added to its corner orientation value. The bottom two corners act in the same fashion, so the net change in the turniness is 1+2+1+2, which is 0 mod 3. Therefore, the turniness is invariant under legal moves.

The fact that parity was conserved implied that only half of the positions counted in equation (1) could be reached. The same reasoning applied to turniness drops the number of attainable positions by a factor of three, leaving one sixth of the counted permutations still possible.

"Flippiness" is for the edges what turniness was for the corners. Each edge is assigned an orientation value of 1 or 0, and then the flippiness is defined to be the sum of these twelve values, mod 2. The value of every edge is 0 when the Cube is solved. Fig. 5 shows how a preferred





ufr: 0
urb: 0
frd: 0
rdb: 0

sum is 0 mod3

ufr: 1
urb: 2
frd: 1
rdb: 2

sum is 0 mod3

fig. 4) The change in corner orientation values after one move.

direction of rotation can be defined on all three planes which cut through the center of the Cube (right hand rule on top, front, and right directions), allowing an edge cubie's orientation to be compared with its solved orientation. If the edge's orientation is the same as in the solved position, then its value is 0; otherwise, the orientation is defined to be equal to 1.

Once more the effect of a single move is studied. Consider a pair of opposite edges on the side to be moved. Before the side is turned, the orientation of the edges is

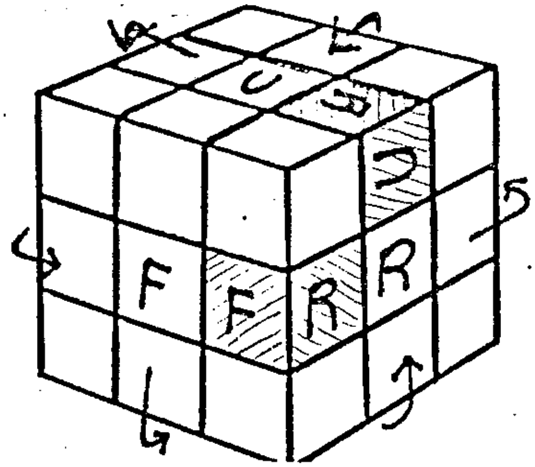


fig. 5) Bands defining orientation of edges. Use right hand rule, with thumb along u, f, or r. Edge fr has value 0 here; ru has 1.

defined relative to some direction, and after the move the standard is some new direction. But both edge orientation values are defined relative to the same direction; therefore, either both the orientations stay the same after the move, or both flip. In either case, the flippiness remains the same, since either 0 or 2 (which is 0 mod 2) is added to it for each pair of opposite edges moved. Fig. 6 illustrates this.

Again a constraint on the number of possible positions has been found. Since the starting position of the Cube has flippiness 0 and no legal move can change the value of the flippiness, all positions derived from a solid faced Cube have flippiness 0. This eliminates another half of the positions counted in equation (1).

These three invariants are the only limitations on the positions of the corners and the edges. Therefore, the

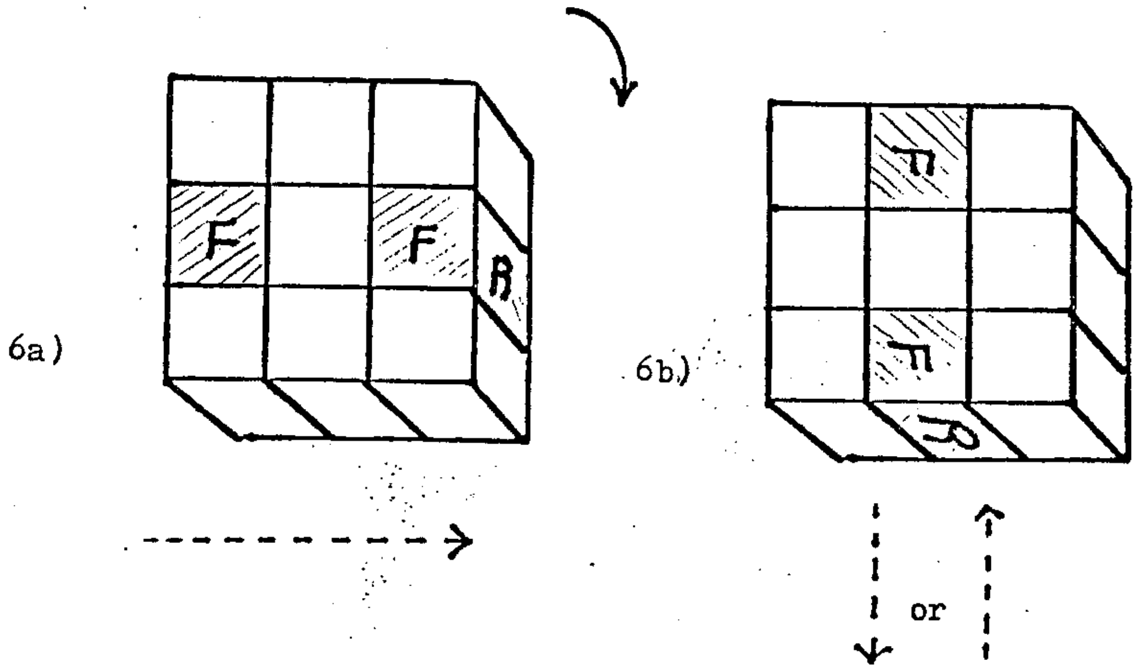


fig. 6 a) Two edges and direction defining orientation.
b) After one move, both edges have the same orientation or both flip.

true value for the total number of positions is

(2) $8!\ 12! \cdot 3^7 \cdot 2^{10}$ = total number of positions, 3x3x3 S-view Cube

$= 4.3 \times 10^{19}$

Later, in the section concerned with solving the Cube, algorithms for moving to all possible positions will be given; for now, it will be assumed that any position can be attained as long as the flippiness, turniness, and parity are the same in the initial and final position. In other words, it is not possible to swap two cubies or rotate one cubie in place without disturbing the rest of the Cube. Thinking of these three restrictions as conservation laws leads to a concise summary of this theory:

3x3x3 Cube, S-view (only edges and corners move)

First Law: Cubies may be grouped into non-interacting types.

Second Law: Any legal sequence of moves conserves the parity, turniness, and flippiness of the Cube.

The resemblance to a theory of physics is not coincidental. The object here is to understand which permutations of the corners and edges are not possible positions; the object of physics is to understand some physical system. Of course, the fact that the author is a student of physics may not be inconsequential.

The next logical step would be to generalize this theory to the 4x4x4 or even an NxNxN Cube; however, at this point the S-view runs into problems. First, the 4x4x4 Cube does not have any fixed cubies, so a co-ordinate system based on the side colors cannot be set up. Second, intuitively it seems that the two center planes should be able to rotate without moving the outsides. This will kill the parity argument as stated, since the "inner" moves cycle four of the twenty-four edges but do not disturb the outer eight corners. What about the new face centers? Should they be returned to their original orientation when the Cube is solved, or should they just be on the correct side?

These problems will just get worse if larger size Cubes are studied using the S-view. Fixing attention only on the colors on the outside of the Cube is simply too inelegant. Another point of view must be found.

- the NxNxN (O-view)

Think of the 3x3x3 Cube as an array of 27 identical cubies, each a solid cube. Only the outside faces show, but all have colors on six sides, just as the solved Cube does. One cubie, in the exact center of the Cube, cannot be seen. Fix an x-y-z Cartesian co-ordinate system in space so that the y axes points "up," the same as the t face of the Cube in solved position, and the z and x axes align with the f and r faces, as shown in fig. 7. The Cube will be considered to be in its "solved" or starting position when all 27 cubies have their r,t, and f faces along the x,y, and z axes. One legal move consists of a quarter turn of any plane of the Cube. Note that it is now possible to make a move which changes the edges without disturbing the corners by rotating the center plane of the Cube. This way of looking at the Cube will be called the O-view, or orientation view since it focuses attention on the spacial

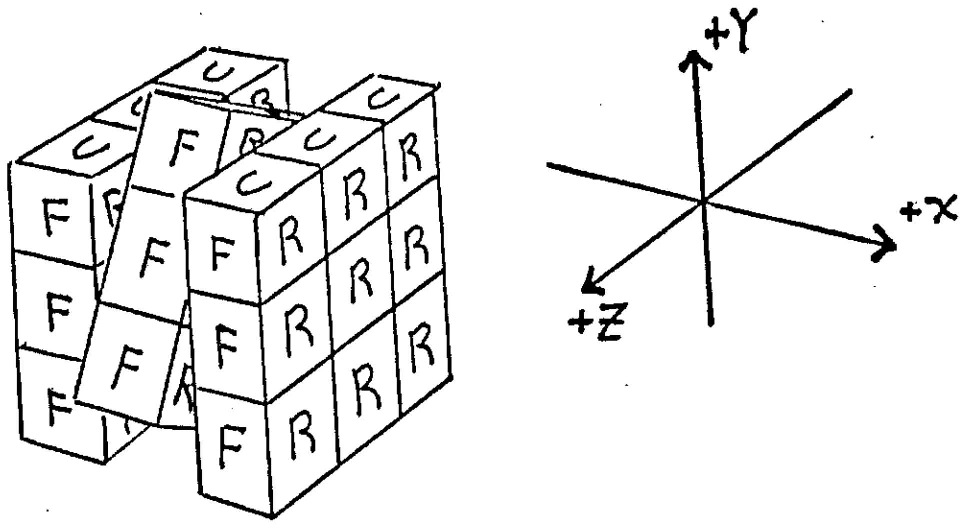


fig. 7) The 3x3x3 O-view Cube

orientations of the cubies.

What constitutes a "position" has been redefined, so of course the number of possible positions has changed. In particular, changing the spatial orientation of the entire Cube or rotating any of the center faces will not change the colors on the outside of the Cube; therefore, all these different O-positions correspond to a single S-position. A new theory of positions is needed. However, rather than modifying the existing 3x3x3 S-view theory, a more general theory of the NxNxN Cube will be built up from the basics. The 3x3x3 will then be just a special case.

The approach will be recursive- assume the (N-2)x(N-2)x(N-2) Cube is understood, then do the NxNxN by adding another layer of cubies on the outside. Note that the next layer of cubies inside an odd Cube is also odd;

therefore, the Cube inside the NxNxN is not the (N-1)x(N-1)x(N-1).

As before, the first order of business is to separate the cubies into different types. A more precise definition of "type" is required than the handwaving of the last section. Any group of cubies will be said to be of the same type if they can be moved to each other's cubicles. For the 3x3x3, this implies that there are exactly 4 types of cubies: 1 central cubie, 6 center face cubies, 8 corners, and 12 edges. Notice that adding another layer of cubies on the outside to make a 5x5x5 will create some new types (another kind of corner, for example), but does not affect the inner types.

The natural approach would be to find out how many different types of cubies there are on the outside layer of an NxNxN Cube, and to examine how each behaves. An appropriate invariant might be defined on each type, as on the 3x3x3's edges and corners (for a geometric version of this procedure on a 7x7x7, see fig. 8). After all this work, an intriguing pattern would emerge: each cubie type either acts like one of those already seen on the 3x3x3, or contains exactly 24 cubies, each with a single orientation per cubicle. Remember, the corners and edges of the 3x3x3 had two and three orientations per location; why now only a single orientation?

Consider a single solid cubie sitting in 3-space. There are 24 ways this cubie can be oriented relative to a fixed set of Cartesian co-ordinates (any one of 6 faces along the +z axis, then one of 4 faces along the +x). Now put that cubie inside a larger Cube, and notice that it will still have 24 possible spatial orientations. And, each of these orientations will always correspond to the same cubicle. Rather than visualizing one move as a rotation of a slice of the Cube, look from the cubie`s point of view. As far as it can tell, the entire Cube is rotating. This perspective makes the argument easy to follow.

Corner cubies have 8 possible locations each, and 3 orientations per location, or 8 x 3 = 24 total possible orientations. 3x3x3 edge cubies have 12 possible locations and 2 orientations per, or 12 x 2 = 24 total possible orientations. Center face cubies have 6 possible locations and 4 orientations per, or 6 x 4 = 24 total possible orientations. By now it is not too hard to see why a type which has 24 cubies only allows 1 orientation per cubicle.

The reason for turning to the O-view of the Cube is becoming clearer. In this viewpoint, the "orientation" of a cubie will always refer to one of its 24 possible alignments with the xyz axes. This orientation determines the position of the cubie on the Cube uniquely, given its type. Usually, people unconsciously make a distinction between the location of a cubie on the Cube and its "orientation" in that cubicle. Given the prevalent S-view of the Cube, this distinction is understandable; however, it is actually misleading and inelegant.

Before continuing, some new terminology is required. The cubies added to an NxNxN Cube to get an (N+2)x(N+2)x(N+2) Cube will be called the N+2 "layer." Thus the 5x5x5 Cube contains a 1 layer (the single central cubie), a 3 layer (the 26 cubies visible on the 3x3x3 Cube), and a 5 layer (the 98 outside cubies). Two types which act in the same fashion belong to the same "class." "Corner," for example, is really a class of cubies, not a type, since 3x3x3 corners and 5x5x5 corners are different types.

All the different classes of cubies on the outer layer of a 7x7x7 Cube are shown in fig. 8. Adding more layers will not introduce new classes, only more examples of these. Once the properties of each class are understood, the number of types of each class on the N layer can be counted, and the number of positions of each type multiplied together. Then, after considering constraints like parity and including the different positions the inner layers can attain, the total number of possible positions of the NxNxN Cube will have been found.

fig. 8) the seven different classes on the 7-layer

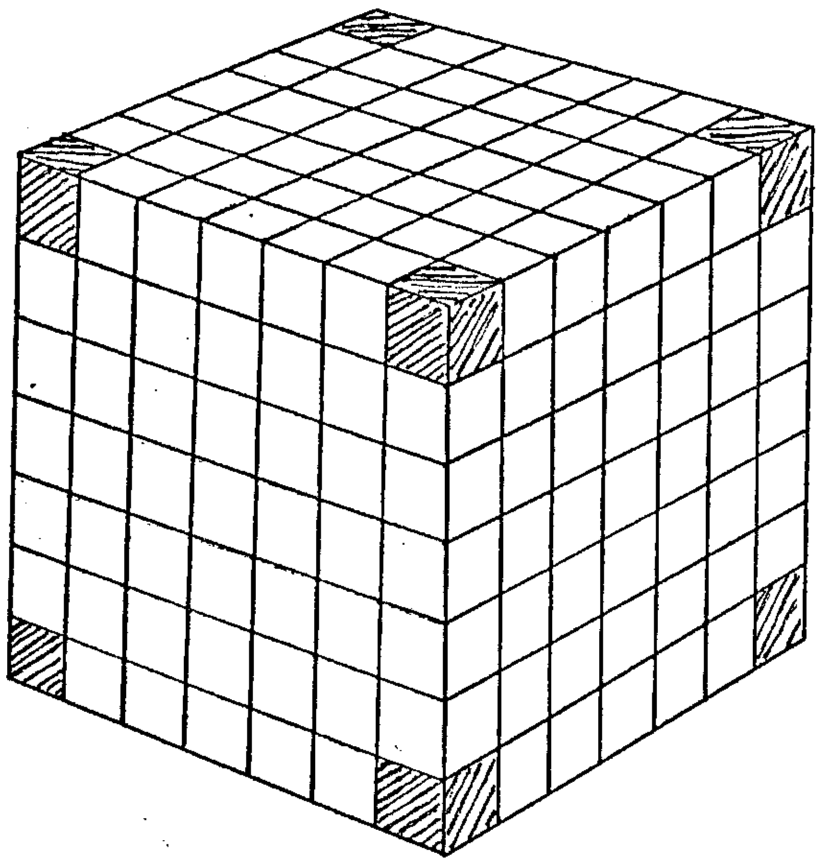


fig. 8a) corner

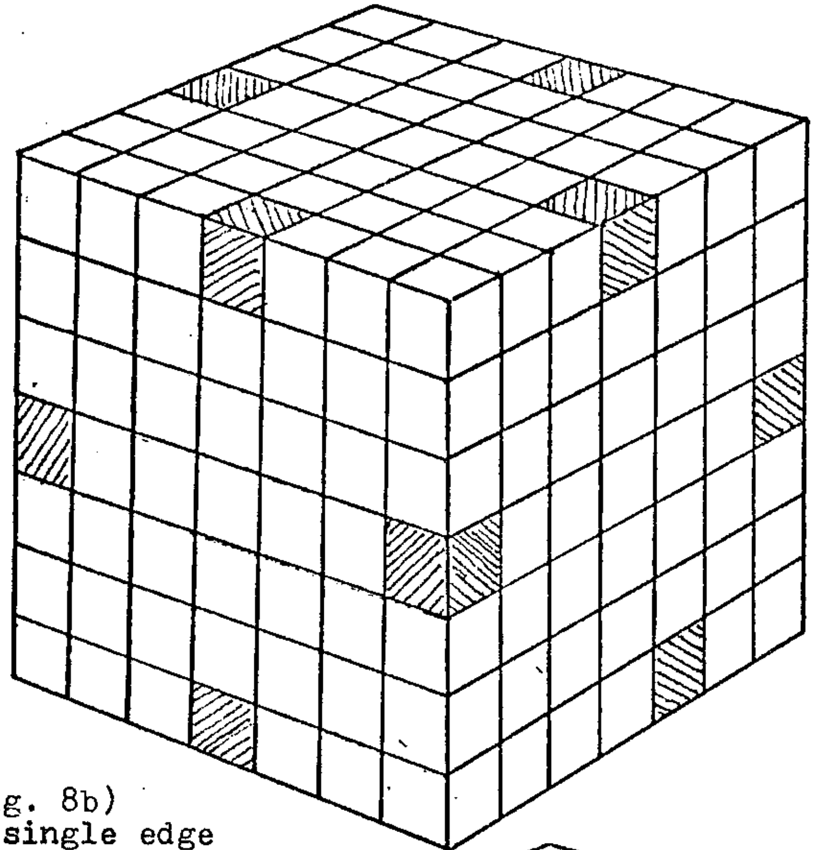
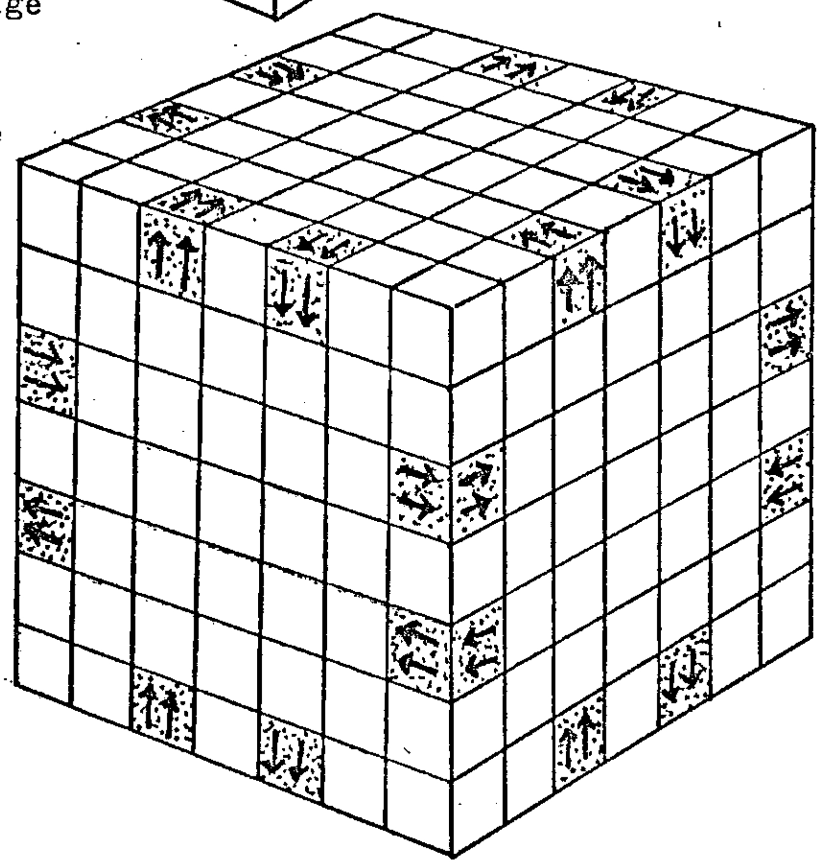


fig. 8b)
single edge

fig. 8c)
double edge



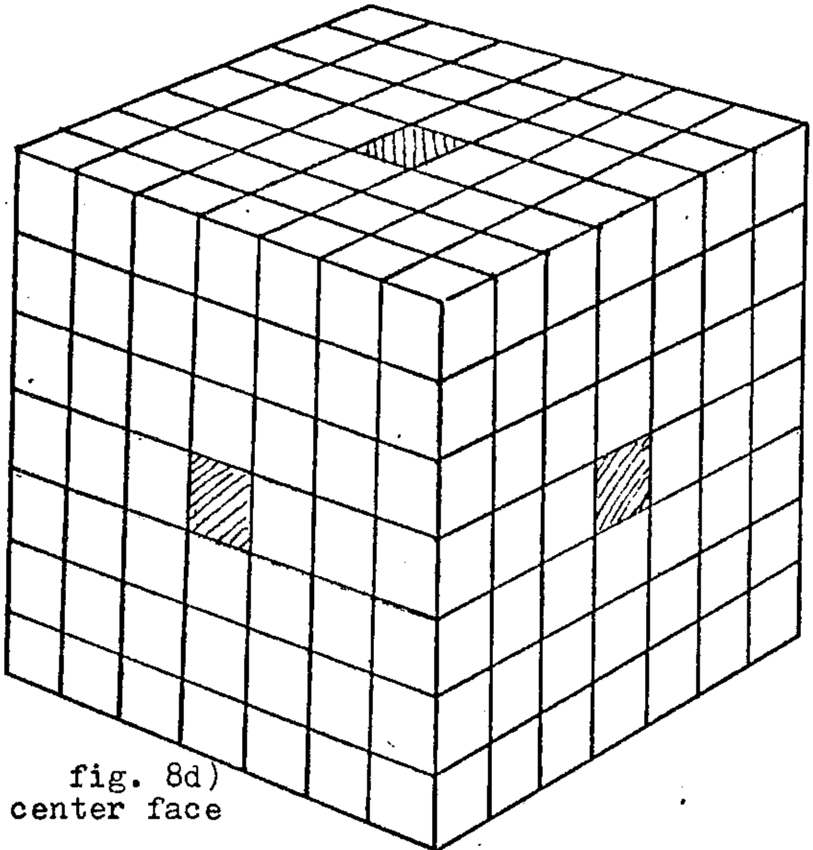
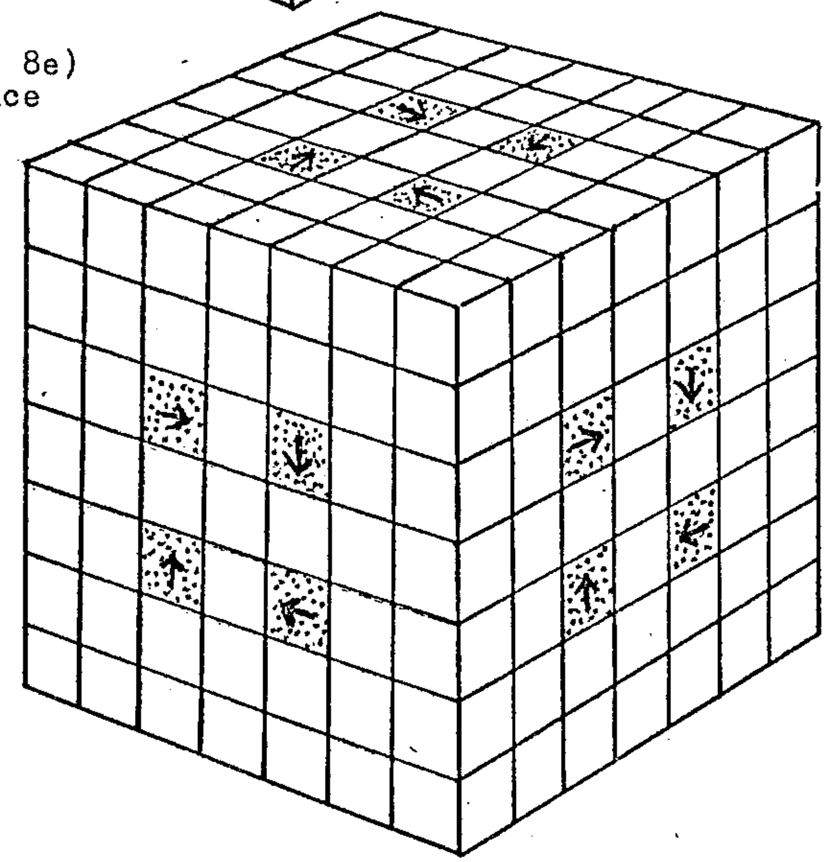


fig. 8d)
center face

fig. 8e)
corner face



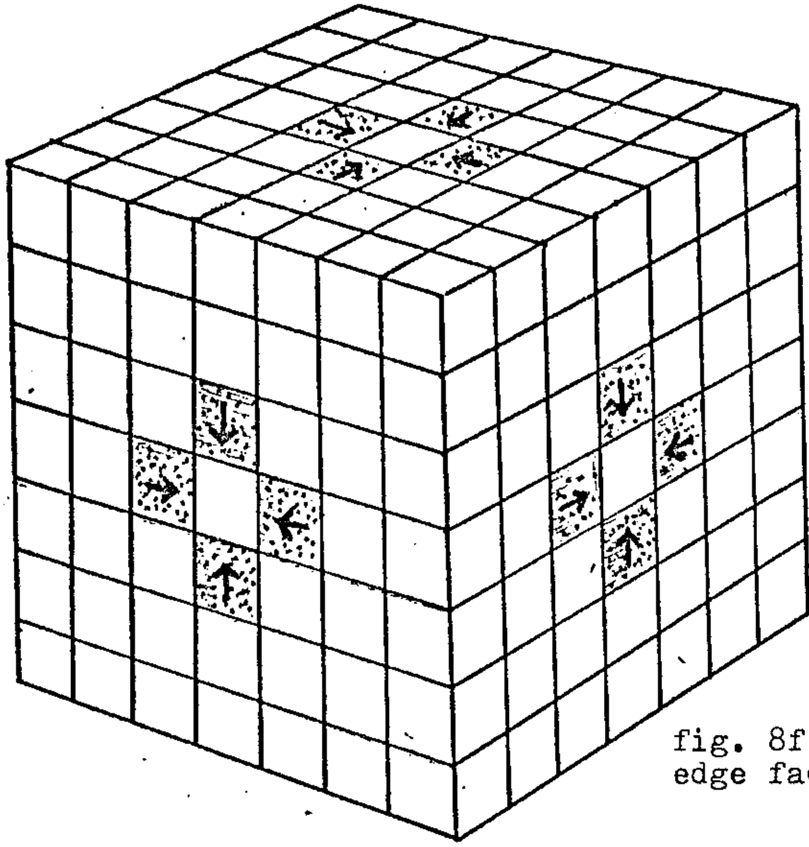
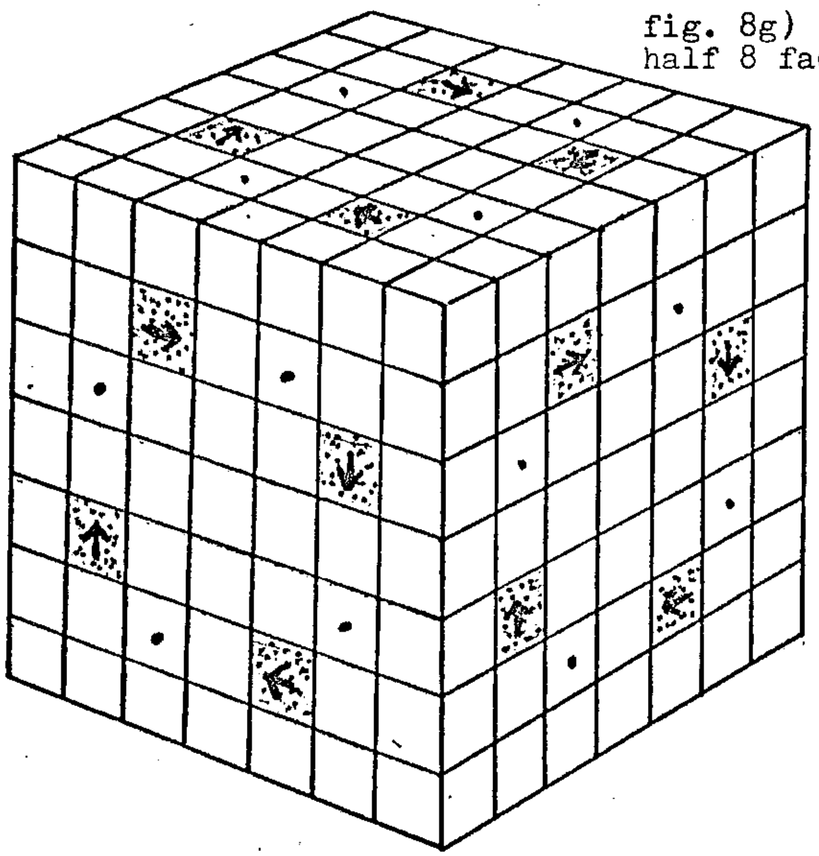


fig. 8f)
edge face

fig. 8g)
half 8 face



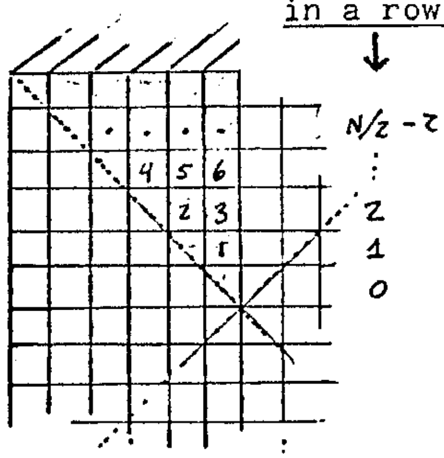
Just eight classes are enough to capture all the cubie types on an NxNxN Cube. The first is the 1x1x1 "central" cubie found in the center of all odd Cubes. This is the only class not displayed in fig. 8. A central cubie always remains in the same cubicle, where it has 24 orientations. The second class is "corner," which has already been seen on the 3x3x3 Cube. Every layer except the 1 layer has exactly one corner type, with 8 cubies and 3 orientations per cubicle. Edges come in two classes, "single" and "double." If a type has 12 cubies, each located in the middle of an edge on an odd layer, then it is a "single edge." Any single edge cubie (like the ones on the 3x3x3) has 2 orientations per cubicle. Other edge types are "double edges," having 2 of their 24 cubies on the same edge.

Types with 24 different cubies are marked with arrows in fig. 8, to illustrate that only one orientation is possible per location. No sequence of moves will change the the arrows, which may be thought of as invariants constraining the possible orientations.

The only classes left are those on the faces. In the middle of every face on an odd layer is a "center face" cubie, as on the 3x3x3. Each type has 6 cubies, with 4 orientations per cubicle. Notice that fixing the orientation of the central cubie keeps each center face cubie in a single cubicle. The "corner face" and "edge face" classes are easily understood; their cubies lie along the diagonals and central column/files of a face. Each type has 24 cubies, 1 orientation per cubicle. The last class is the "half 8 face." This peculiar name is given because at first glance it seems that 8 cubies on a face should belong to a single type. Actually, only half of them are one type. These types also have 24 cubies each.

Turniness and flippiness must now be defined relative to the xyz axes rather than the face centers, but after this is done the same result found in the last section applies: each corner and single edge type must conserve these variables. The central cubie determines the locations of the face centers. All other types, taken separately, are free to be in any permutation. Note here how powerful the O-view is. Each of the new cubies has 24 possible locations, orientation determines location, and there are 24 different orientations; therefore, each cubie like this has one orientation per location.

How many types of each class are there on the N layer? Counting them is straightforward. The only tricky calculation is computing the number of half 8 face types, which is shown in fig. 9a. How many permutations for each class? The number of possible corner and single edge positions with constant flippiness and turniness has already been done. The central cubie has 24 positions, and each face center has one of four orientations in a determined cubicle given the central cubie`s status. The other types have 24 cubies; therefore, each has 24! permutations. These number of half 8 types in a row



total =

$$2(1+2+\ldots+(N-4)/2)$$
$$= (N-2)(N-4)/4$$

fig. 9a) Counting the number of half 8 types on the N layer.

fig. 9b) Properties of the eight classes.

class	number of types N odd (not 1)	N even	number of positions per type
corner	1	1	$3^7 \cdot 8!$
single edge	1	0	$2^{11} \cdot 12!$
double edge	$(N-3)/2$	$(N-2)/2$	$24!$
center face	1	0	4^6
corner face	$(N-3)/2$	$(N-2)/2$	$24!$
edge face	$(N-3)/2$	0	$24!$
half 8 face	$(N-3)\cdot(N-5)/4$	$(N-2)\cdot(N-4)/4$	$24!$

characteristics are listed in fig. 9b.

Now that each type's individual behavior is understood, the constraints on the interactions between types must be explored. When the 3x3x3 Cube was done, it was seen that the parity of the edges and corners had to match. The same result holds for the NxNxN, but now the parity of all the types of cubies must be consistent. Again, the parity of a type is defined as 0 (or "even") if the permutation of the cubies is even, and 1 (or "odd") if the permutation is odd.

Let any move which moves the corners on the N layer be called an "N-move." The central slice move on an odd Cube will be called a 1-move. Notice that every move affects exactly one type of corner or the central cubie, so the parity of the corners and the central cubie can be used to define the parity of the entire Cube. The parity of every other type must agree with the parity of the corners and central cubie. This is the only constraint on the interactions of the cubie types.

For example, move the center plane of the 3x3x3 one quarter turn. The parity of the central cubie is now 1, and the parity of the 3-corners is 0 (still in solved position). Any possible position must have the parity of the edges consistent with this, i.e. the edge parity must be 1.

Finally the point has been reached when all these results may be put together. Fix all the layers below N in some given position, and keep the parity of the N-corners at 0. For each type on the N-layer, half of the positions counted in fig. 9b are consistent with this (since for every m, only an even number m-moves are allowed in moving to a new position). Define P(N) to be the number of positions which the N-layer can attain, keeping the inner layers fixed. One half of these keep the N-corner parity at 0, so

$$(3) \qquad P(N)/2 = \prod_{\substack{\text{types on} \\ \text{N layer}}} (\text{1/2 number of positions for each type})$$

$= 24/2$ if $N=1$

$= (3^7 \cdot 8!/2)(2^{11} \cdot 12!/2)(4^6/2)(24!/2)^{3(N-3)/2+(N-3)(N-5)/4}$

if N odd, $\neq 1$

$= (3^7 \cdot 8!/2)(24!/2)^{2(N-2)/2+(N-2)(N-4)/4}$

if N even

and the total number of positions of the NxNxN Cube, defined to be T(N), is

$$(4) \qquad T(N) = \prod_{i \text{ odd } N} P(i) \qquad \text{for an odd Cube}$$

(For N even, product is over even values of i.)

Subbing in and collecting terms gives

(5)

$$\ln T(n) = \ln 24 + \left[(N-1)/2\right]\cdot\ln(8!\,12!\,3^7 2^{21}) + \left[\sum_{\substack{i=1\\ \text{odd}}}^{N} \frac{(i-3)(i-1)}{4}\right]\cdot\ln(24!/2) \quad \text{if } N \text{ is odd, and}$$

$$= (N-2)\cdot\ln(8!\,3^7) + \left[\sum_{\substack{i=2\\ \text{even}}}^{N} (i-2)(i+2)/4\right]\cdot\ln(24!/2) \quad \text{if } N \text{ is even.}$$

$$\Downarrow$$

$$T(N_{odd}) = 24(8!\,12!\,2^{21}3^7)^{(N-1)/2}\,(24!/2)^{(N^3-13N+12)/24}$$

$$T(N_{even}) = (8!\,3^7)^{N/2}\,(24!/2)^{(N^3-4N)/24}$$

The summations can be done by fitting a cubic to the function sum=f(N). There is an easier way to understand the basics of this formula, though. All the classes of types of cubies (whew!) divide into three sets: a) the central cubie, b) the corners, single edges, and center faces, and c) all types having 24 cubies. An odd NxNxN Cube contains exactly 1 central cubie, and (N-1)/2 types of corners, center faces, and single edges. That accounts for 1+(8+6+12)(N-1)/2 of the cubies; all the other types have 24 cubies, so there must be [N**3 - (1+13*(N-1))]/24 of them. But this last expression is just (N**3 - 13N + 12)/24, which is the exponent on the last term in (5). So T(N) is just the product of the number of positions of each type, each with an exponent counting how may of that type there are.

Collecting terms of the same order, T(N) can be re-written as

(6)

$$T(N_{odd}) = \left[24(8!12!2^{21}3^{7})^{-1/2}(24!/2)^{1/2}\right]$$

$$\cdot\left[(8!12!2^{21}3^{7})^{1/2}(24!/2)^{-13/24}\right]^{N}$$

$$\cdot\left[(24!/2)^{1/24}\right]^{N^3}$$

$$= (44.9)(0.0561)^{N}(9.52)^{N^3}$$

$$T(N_{even}) = \left[(8!3^{7})^{1/2}(24!/2)^{-1/6}\right]^{N}\cdot\left[(24!/2)^{1/24}\right]^{N^3}$$

$$= (1.14)^{N}(9.52)^{N^3}$$

For large N, T(N) is about 9.5**(N**3); a particularly simple result. This may be interpreted as saying that for the purposes of counting positions, each cubie on a large Cube acts as if it has nearly 10 states, independent from the rest of the Cube.

Needless to say, this function T(N) gets very big very fast. Fig. 10 shows a plot of log(T) vs. N. In spite of the fact that the derived formula for T(N) has a different form for odd and even N, the points do lie on a smooth curve.

The next problem to be tackled on the road to understanding the Cube is how to manipulate it into these different positions. What sequence of moves will take the Cube from state A to state B? But before ending this section, it seems appropriate to again sum up the basic laws on which O-theory of the positions of the Cube rests. Some of these ideas will continue to be useful in the next perspective on the Cube.

NxNxN Cube, O-view

(orientation of all cubies noticed)

First Law: Cubies may be grouped into different types. which interact only through parity consistency.

Second Law: Flippiness and turniness are conserved.

Third Law: Every cubie has exactly 24 different orientations.

Fourth Law: Each orientation corresponds to a unique location.

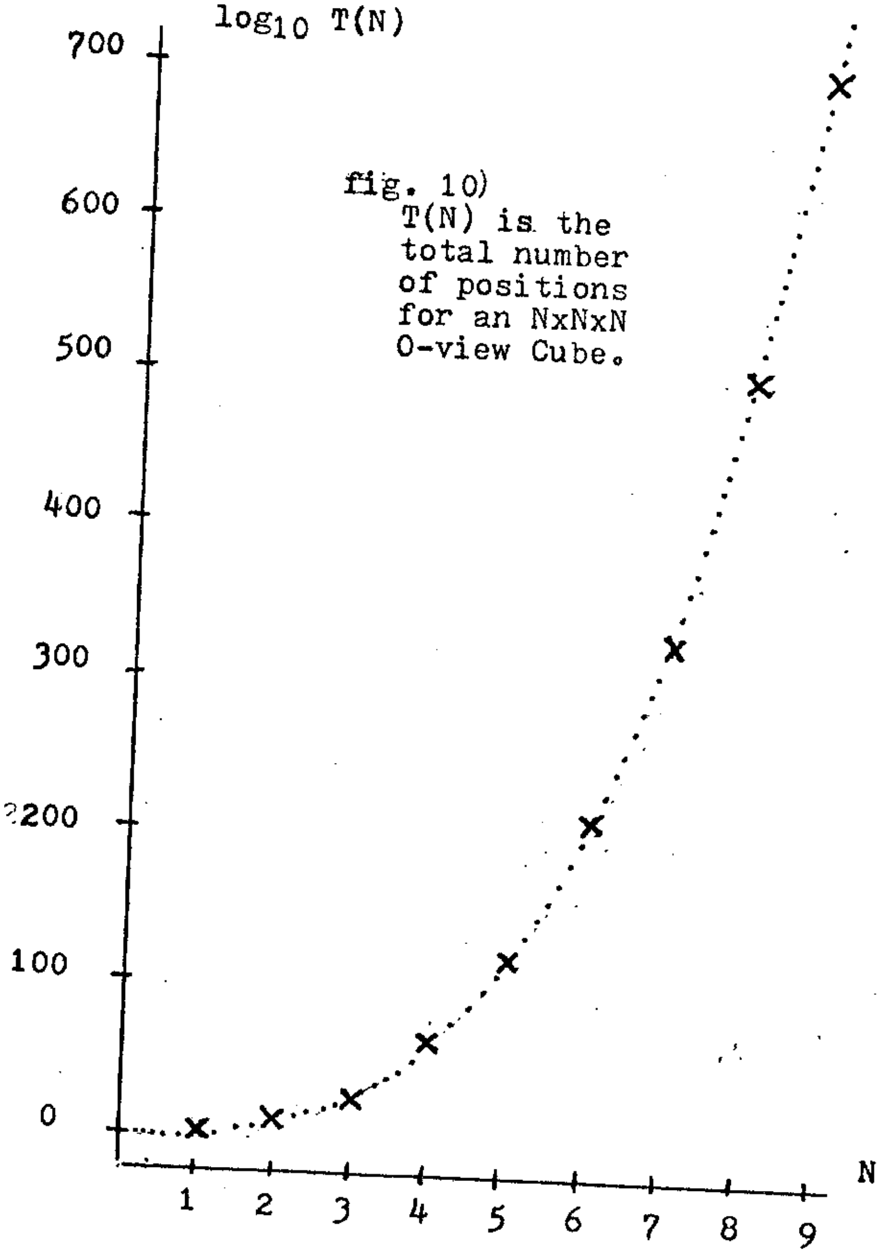


fig. 10)
T(N) is the total number of positions for an NxNxN 0-view Cube.

III. Solutions

The first thing people ask when they see a Cube is "Can you do it?" Clearly, a full understanding of the Cube would enable a person to manipulate the Cube to any possible position, but this knowledge alone is not enough. As can be seen from the last section, the more one learns, the more there is to understand.

Anyone reading this chapter hoping to find a quick and dirty way to solve the Cube will probably be disappointed. Holding a 3x3x3 Cube in your hand and turning to change the colors is more of an art than a science; a discipline which can only be partly explained. Instead, some basic principles of transformations will be discussed, and a few easily explained but incredibly long algorithms will be described, all in the context of group theory.

- Group Theory

In the last chapter, invariants were used to find the restrictions on the number of possible positions. A new problem is being considered now; therefore, the approach must be slightly different. The new point of view will be group theory, the same mathematical theory that describes particle physics, quantum mechanics, and cosmology.

A mathematical group consists of a set of elements, usually called operators, and a binary product. The elements of the Cube group are the possible positions of the Cube, and the product AB of two positions A and B is the position reached after doing first A, and then B to the solved Cube. More precisely stated, if abc...d is a sequence of moves from the solved position to position A, and mno...p goes from solved to B, then AB is the position reach after the moves abc..dmno...p have been applied to the solved position. On the 3x3x3 S-view Cube group. the twelve possible single moves (U,U',D.D',R,R',L,L',F,F'.B,B') generate the rest of the group. The identity element I is always the solved position. Note that any element A which can be described by the move sequence abc...de has an inverse called A' equal to e'd'...c'b'a'.

Now that the Cube has been modeled as a group. all the theorems, notions, and language (i.e. the specific world view) of group theory works on the Cube. This is exactly the power of mathematical modeling. The next step is to take some of the useful ideas of group theory, namely subgroups. permutations and cycles, adjoints, and commutators, and translate them into forms appropriate for the Cube.

A subgroup is a piece of a group which is closed and contains the identity. Usually subgroups are found by fixing attention on a specific feature of the larger group. Some of the subgroups of the Cube have already been discussed, but before they were called "types" (to make this precise requires equivalence classes and mods, but that's not the point here). The types are essentially independent; therefore, if these subgroups can be solved individually, the Cube will be solved. The other interesting subgroups of the Cube are those elements generated by some smaller than complete set of operators. Singmaster has a nice discussion of these in his article.

A permutation group is a set of operators which move around m objects in n boxes. Ignoring for a moment the fact that some cubies have more than one orientation per cubicle, the different positions are just different arrangements of cubies in cubicles. An important (here, anyway) theorem in group theory claims that any permutation of a finite number of objects can be broken into disjoint cycles, where a cycle takes n objects and shifts them sequentially in a circle (a to b`s place, b to c`s, ... n to a`s place). This theorem will come in handy later in the search for useful operators.

Given operators A and B, any operator of the form BAB' is called an adjoint of A. People who can solve the Cube usually have a fairly small set of operators which they know cold, sequences of moves which will change small, specific parts of the Cube. However, they hardly ever use exactly these operators, they use their adjoints. Hofstader talks about this in his article. For example, say Joe has a sequence of moves (operator A) which will cycle three corners on the top of the Cube (those in cubicles ufl, ufr, and ubr), without disturbing anything else. Assume Joe wants to cycle two cubies on the top layer and one on the bottom (ufl, ufr, and dbr cubicles). What he does is 1) move the one on the bottom to the top with the quarter turn B, then

2) do the operator A. and finally 3) move the out of place slice back via B'. This sequence is just an adjoint of A, namely BAB'.

The last group theory term to be defined is the commutator of two elements A and B, by definition equal to ABA'B'. In quantum theory, the non-vanishing commutator of position and momentum gives rise to the uncertainty principle. Here also, the commutator gets much of the interesting nonintuitive behavior. If the group were Abelian (or commutative. AB=BA), then ABA'B' would just be AA'BB'=I. The Cube, however, is not even close to Abelian, so that even though most of the cubies are restored after ABA'B', not all are. It will turn out that a thorough understanding of this kind of idea and a smattering of Cube sense is enough to solve the NxNxN.

Enough background. On with solving the Cube.

- a 3x3x3 method

The procedure described here to solve the 3x3x3 S-view Cube is incredibly slow, but does have a few interesting features. First, all the operators are variations on a simple four move commutator. Second, finding the corner operator requires a trick which can be used quite generally. Last, and most important, this algorithm isn't too messy to explain.

The recipe contains the following steps: 1) get the parity equal to 0; 2) put the edges in the right places; 3) orient the edges; 4) put the corners in the correct cubicles; 5) orient the corners. Clearly, when all these have been done, the Cube will be solved. Each will be taken in succession.

1) is simple; just make any one move if the parity is 1, and do nothing if the parity is already 0. If at some later stage something goes wrong, reset the parity to 0 and start again.

2) is done by 3 cycles on the edges, letting the corners move anywhere they will. FR'F'R cycles the three edges around the ufr corner without moving any other edges, and FRF'R' cycles 3 edges in a straight line. A single path connecting all the edges can be made using these triangles and lines. Fig. 11 illustrates these ideas. Put each edge in its proper cubicle by starting at one end of the path and working down, moving edges only along the path via the operators described. This procedure only fails if at the end two edges need to be exchanged; however, this would mean that the parity was not 0.

3) is tedious. too. Examine the commutator shown is fig. 11a, FR'F'R. This disturbs only three edges and four corners. the ones around the ufr corner. The commutator of R and U' (RU'R'U), or of U and F' (UF'U'F) move the same three edges and four corners. After all three have been done in succession (the sequence [FR'F'R][RU'R'U][UF'U'F]), all the cubies have been restored to their original cubicles. but

fig. 11a)

FR'F'R

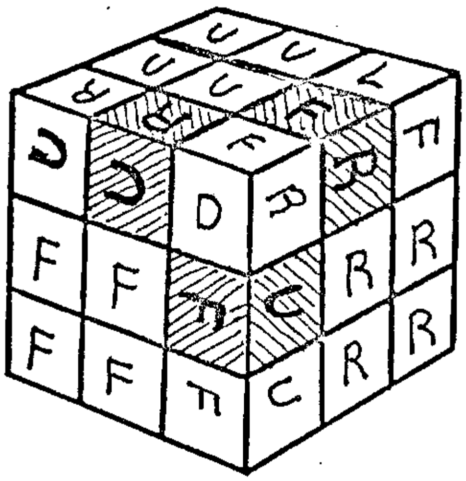


fig. 11b)

F R F'R'

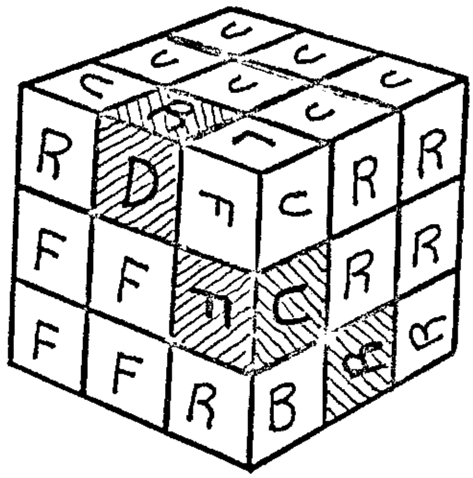


fig. 11)
Placing
the edges.

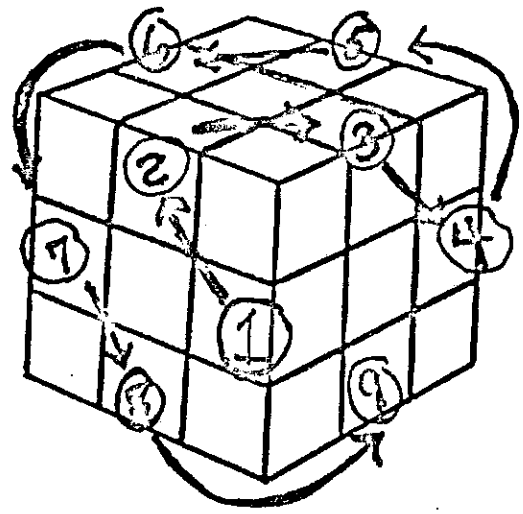


fig. 11c)
a path

three corners are turned and two edges are flipped. The position is shown in fig. 12. To orient the edges, just apply this routine as often as necessary along the path, starting at one end and working down as before. Using adjoints of this operator will probably shorten the required number of applications.

4) uses the trick mentioned earlier. The idea is

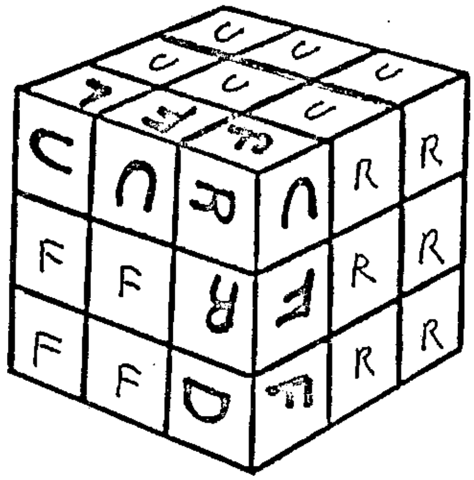


fig. 12)
the orienter

(F R'F'R)(R U'R'U)(U F'U'F)

fairly simple: take any reasonably short sequence of moves and look at the resulting position as a set of independent cycles. The theorem of the last section as witness that this is possible. The length of the cycles may be relatively prime; if so. repeating the sequence enough will restore some but not all of the cycles. The result may be that only a small number of cubies have moved. With luck, a useful operator has been found. Note that if lcm is the least common multiple of the cycle lengths, then repeating the sequence lcm times should restore the Cube.

To apply this to the corners on the 3x3x3, examine the sequence FRF'R' (fig. 11b). This is one 3-cycle on the edges and two 2-cycles on the corners (ignoring corner orientation). Repeating the sequence three times restores the edges, disturbing only two pair of corners. Moving the top slice, doing the sequence of moves another three times, and restoring the top slice gives a 3-cycle on the corners which does not move anything else. A path, like the one on the edges, can be found connecting the corners. Moving corners along the path using the 3-cycle will then restore

fig. 13a)
$(F\ R\ F'\ R')^3$

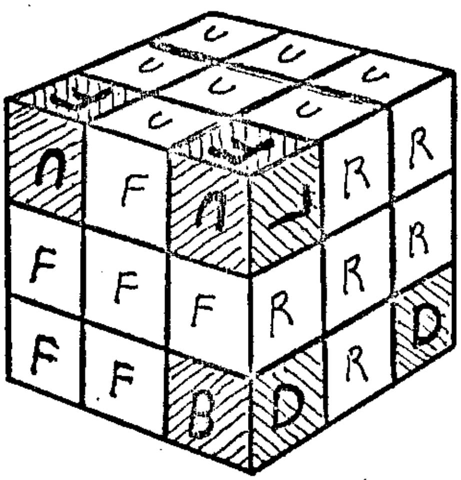


fig. 13b)
$(F\ R\ F'\ R')^3U(F\ R\ F'\ R')^3U'$

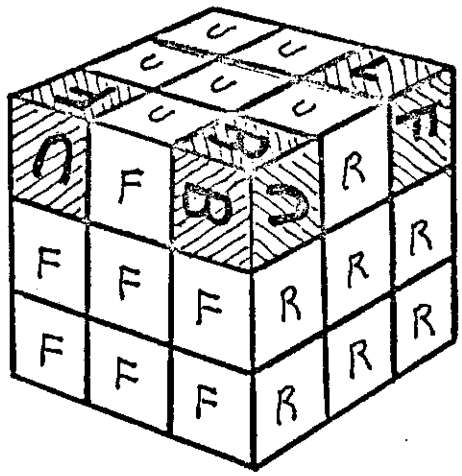


fig. 13)
Placing
the corners.

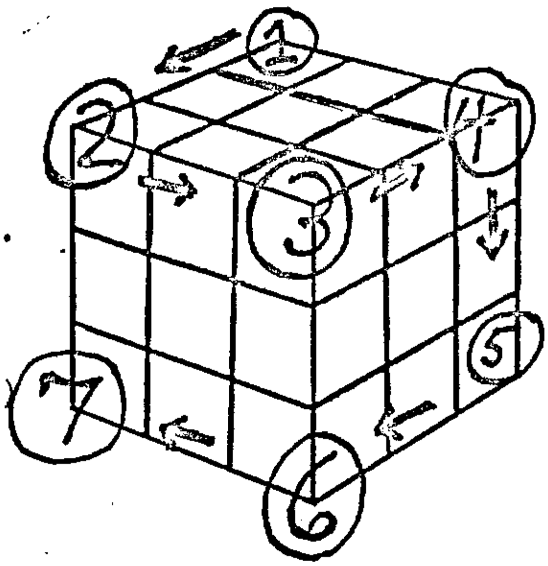


fig. 13c)
a path

each to its proper cubicle. All this is shown in fig. 13.

5) is the last step in this 3x3x3 solution recipe, the orientation of the corners. The same operator that oriented the edges will work again (see fig. 12), just do it twice so that it leaves the edges unaffected. This gives a sequence which turns three corners on the same layer in the same direction. By turning corners a, b, and c clockwise and then turning a, b, and d counter-clockwise, c and d can be turned in opposite directions without affecting anything else. These operators are enough to orient the corners.

Voila! A procedure for solving the 3x3x3 Cube. However, anyone who actually read through that may not feel particularly enlightened. Think of it as an existence proof, not an algorithm to be learned.

A number of useful 3x3x3 operators are given in A2. Caution, however: without a minimum of Cube sense, i.e. the right point of view, they do not lead directly to a solved Cube. Instead of focusing of the specific move patterns, recognize where the outlook and important concepts have been outlined. A Cubist must see a sequence of moves as an operator, or a "macro" as a programmer might call it, and believe in commutators, adjoints, and subgroups (even if he does not use those words to describe them).

Also, notice that there were three operators which were not searched for: a sequence to swap two corners without disturbing the edges, a single corner orientation operator, and a single edge flipper. Any of these would take the Cube to impossible positions.

- an NxNxN method

As seen in the last chapter, there are only eight different classes of cubies on an NxNxN Cube. Most of these cubies only have a single orientation per cubicle. If operators could be found to manipulate each type individually, then the NxNxN could be solved in the same way the 3x3x3 was. At the end of this procedure , an algorithm would have been found, but not much more would be understood. The whole idea sounds tedious.

There is a better way. "Better" here means more general, simpler, easier to understand, and far more elegant. And in the final analysis, elegance is the only true criterion which distinguishes two working viewpoints. This better way is just a mode of thinking, an outlook which makes moving small numbers of cubies intuitive.

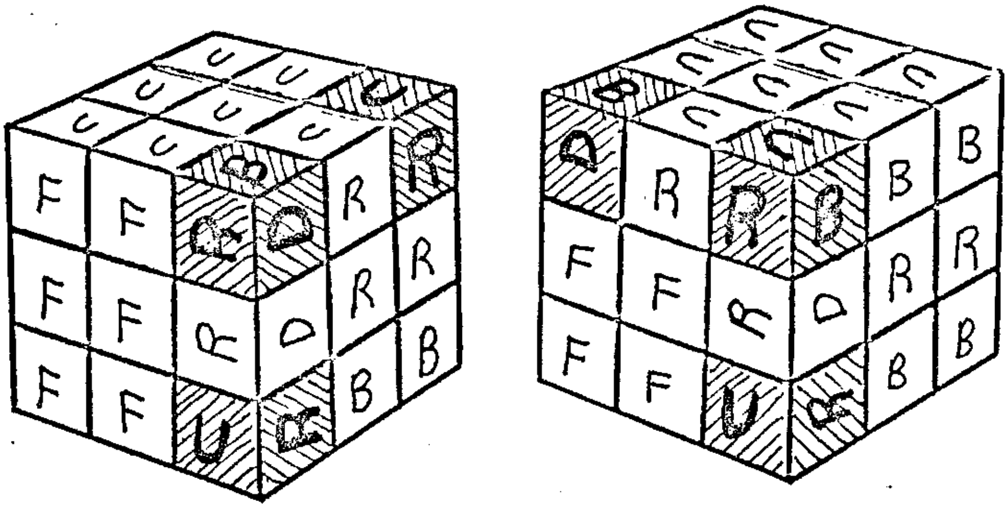
The basics of the Cube solving procedure are still the same: treat each type separately, working from the inner layer to the outer. As each new layer is begun, first set the parity correctly before proceeding, then orient the six center faces if the layer is odd. The only operator needed to finish the layer is one which will cycle three cubies of a given type without disturbing anything else. With a little skill, the corners and single edges can be oriented as they are moved into the proper cubicles; otherwise, they can be turned or flipped by the technique shown in fig. 15.

Here is how the idea works. Pick three cubies (1, 2, 3) to be cycled, 1 and 2 on the same slice S, 3 somewhere else. Find some sequence of moves A which will swap cubies 1 and 3 without affecting the rest of slice S. A can mess up the rest of the Cube, so it can be very short- usually just an adjoint of a slice rotation parallel to S. After doing A, rotate slice S so that cubie 2 is in the cubicle where cubie 1 started. Reverse A, restoring the scrambled parts of the Cube and swapping cubies 1 and 3. Finally, move S back to its original position. The whole sequence is ASA\`S', the commutator of A and S. A simple operator A that randomized much of the the Cube has become a 3-cycle.

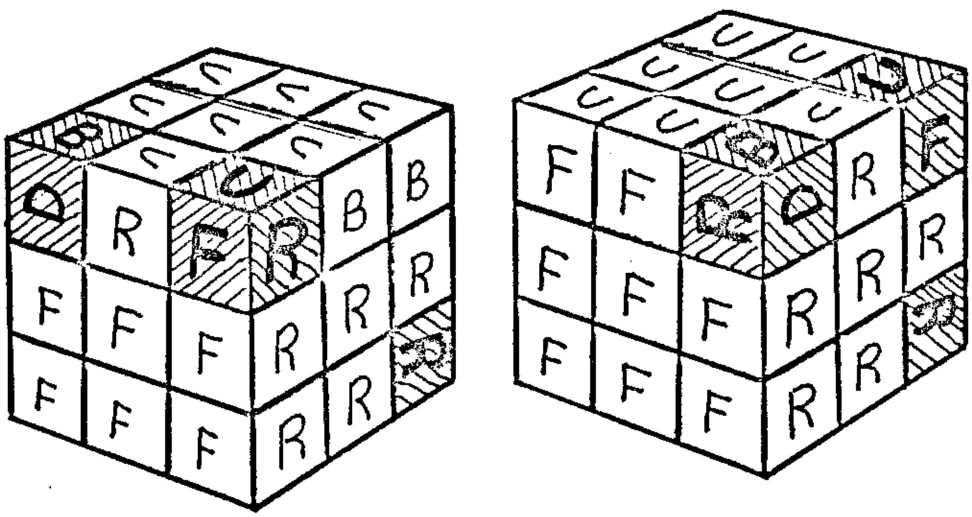
Fig. 14 shows this trick done on the corners of a 3x3x3 Cube. FDF' takes the ufr corner off the top slice, without changing anything else on top. After moving urb over with U, FD'F' will restore the bottom slices and put ufr back on the top slice. The last move is to put the top slice back with U'. Only 3 corners have been moved.

A variation on this technique can be used to flip two edges or turn two corners. by 1) flipping the first edge, leaving only the slice its on unscrambled, 2) moving the slice in order to put another edge in flipping position, 3) reversing the flipper routine, and 4) restoring the slice. Just the same kind of thinking again. David Singmaster described an operator like this in his "Notes," he called it a "monoflipper." The exact sequence to flip two edges on the 3x3x3 is given in A2.

fig. 14) a typical commop, the commutator
of an adjoint of D and the move U.



First do a) F D' F', then b) U,



then c) F D F', and end with d) U'.

Once this way of thinking about moving cubies about on the Cube is understood, then solving the NxNxN becomes intuitive and straightforward. The point is that rather memorizing specific move sequences, the ability to improvise these "commutator operators" (or "commops" for short) should be cultivated, for therein lies true understanding. Anyone who still wishes to see detailed algorithms to solve the NxNxN Cube can turn to the Appendix. They are too tedious to do here.

This entire section has probably been difficult to follow. But whenever a new way of thinking is learned this problem arises. Once the viewpoint is understood, all is well; however, explaining that mind-set can be very difficult. This problem of notation will be examined in the next chapter.

For now, it will be enough to summarize the notions of this section with another law in "the theory of the NxNxN O-view Cube."

NxNxN Cube, O-view
(continued)

Fifth Law: Commutators and adjoints are enough
to manipulate any single type
without disturbing the rest of the Cube.

- How many moves?

When two people who can solve the Cube get together, often the big question each has for the other is "How many moves can you do it in?" Many have worked on the problem; therefore, it will only be mentioned briefly here.

For more information, see David Singmaster's "Notes" or Douglas Hofstader's article.

How can this problem be conceptualized? Imagine that the set of possible positions are points in space, and that positions one move apart are connected by a line. "How many moves?" questions involve trying to recognizably define a metric on this network. The distance between two points can be defined as the length of the shortest path between them, but how can this information be obtained from the two Cube positions? No one has as yet fully answered this question.

An upper bound can be given for the distance "across" the network by considering the worst-case length of algorithms that solve the Cube, since in that number of moves any two points can be connected. The fastest solutions of the 3x3x3 S-view Cube take about 80 moves; therefore, the procedure outlined a few sections back, which could take nearly 1000 moves to finish, does not exactly find a straight path. Perhaps it should be mentioned here again that one S-view move consists of a quarter turn on an outside face on the Cube. Not everyone defines a single move in this way, so care must be taken to avoid confusion.

A lower bound on the size of the network is found by a simple calculation described by Douglas Hofstader. Starting at any point in the network, one move will reach twelve new points, since there are twelve possible moves. Each of these has eleven more (one move from each position goes backward to start) possibly new points another move away, and so on outward. The "edge" of the network will not be reached until, after some number M of moves, the number of points counted exceeds the total possible number of positions; therefore, positions must exist which are at least M moves apart. Doing the math gives

$$(7) \qquad 1 + 12 + 12\cdot 11 + 12\cdot 11^2 + \ldots + 12\cdot 11^{M-1} \gtrsim 4.3\ 10^{19}$$

$$11^M \gtrsim (5/6)\cdot 4.3\cdot 10^{19}$$

$$M \gtrsim \ln(3.6\cdot 10^{19})/\ln(11) \doteq 18.8$$

$$M = 19$$

On the NxNxN O view Cube, one move is a quarter turn of any slice. Since there are N slices in each of three directions, the number of different single moves is 6N. The total number of possible positions, T(N), is still known, so M(N), the same lower bound found above, can again be calculated.

$$(8) \qquad 1 + 6N + 6N\cdot(6N-1) + \ldots + 6N\cdot(6N-1)^{M(N)-1} \geq T(N)$$

$$M(N) \gtrsim \ln T(N)\cdot\frac{3N-1}{3N} \;/\; \ln(6N-1)$$

$$\text{but } T(N) \approx 10^{N^3} \quad \text{for big } N\text{, so}$$

$$M(N) \approx \ln(10^{N^3}) / \ln(6N)$$

$$\propto N^3$$

some sample values

N	T(N)	M(N)
3	$2.1\cdot10^{24}$	20
5	$5.6\cdot10^{117}$	81
10	$2.5\cdot10^{979}$	554
100	$3.6\cdot10^{9.8\cdot10^5}$	352,422

Solving the NxNxN using commops requires about eight moves to orient each cubie, so the maximum number of moves for the entire procedure is of the order 10*N**3 (ten to the n cubed) or so. M(N), the minimum bound on the network, is also nearly proportional to N**3 for large N. Caught between these two, the true maximum number of moves separating two positions must, likewise, grow as the number of cubies. This is important enough to be another law in the Theory of the NxNxN Cube.

The NxNxN Cube, O-view
(continued)

Sixth Law: The size of the space of all possible positions grows as the number of cubies.

All of these solution methods have a fatal flaw. Each aims at subgoals short of a complete solution. None of these "stopping points" need to lie anywhere near the initial and final position; therefore, a solution which passes through them is a long, windy path. There must exist, though, a nearly straight path.

Cubists call this shortest sequence "God`s Algorithm." So far, He has kept it to himself, and He alone knows whether or not its complexity falls in side the human sphere of comprehension. Can even a human/computer team map the wilderness of this Cube network?

Maybe tomorrow some inspired addict will invent a simple, elegant scheme to describe the metric on the space of possible positions, and the long search will be over.

But then again- maybe not.

IV. Representations

Anyone attempting to understand the Cube must find a way to record his work, a language to write and think in. A representation of the Cube in English is unthinkable. English is only truly suited for politics. So far, the only notation that has been used in this paper has been Singmaster's : writing a string of letters to represent a sequence of moves. While this is enough when discussing algorithms, it still does not represent the Cube itself. In particular, given two sequences of moves, the only way to tell if they result in the same position is to actually do them out and then compare the two Cubes. These two strings, for instance, get to the same Cube position.

(9)

a) $R' \ U' \ B^2 \ U^2 \ D^2 \ F' \ U' \ D^2 \ U^2 \ B^2 \ U \ D' \ R \ U$

b) $(U \ F' \ U' \ F^2 \ R' \ F' \ R^2 \ U' \ R' \ U)^3$

Both of these arrive at the same position.

In this chapter, two true representations of the Cube will be presented. Matrices and vectors will be the format of these representations since they are familiar non-Abelian structures which can store a great deal of information. All the properties which have been derived in this paper can be seen in these mathematical languages, for everything that the Cube does, they do also. In addition, any computer program which works on the Cube could well use one of these formats. Each is just another way of looking at the Cube.

-3x3x3 Cube

The S-view of the 3x3x3 Cube notices only the locations and orientations of the eight corners and twelve edges. A straightforward way to represent this information is simply to use a twenty place column vector, where each slot in the vector is a cubicle, the numbers 1- 20 stand for the cubies, and factors of e to some imaginary power represent the orientation. The twelve possible moves can be written as 20x20 matrices which move the numbers in the vector around and multiply them by the appropriate orientation factor. Doing the move U looks like this:

(10)

$$
\begin{bmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix}
\cdot
\begin{bmatrix}
1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20
\end{bmatrix}
$$

Now the Cube has been reduced to a vector which can be transformed by multiplying it by one of twelve matrices. The matrices are not very nice, but at least the entire system is imbedded inside a mathematical system. Unfortunately, understanding the Cube is not made particularly easier by this translation. Nor is it particularly elegant. But then, the S-view never was.

- the NxNxN Cube

Label the cubicles on the NxNxN Cube with vectors from the origin of the xyz co-ordinate system. Rather than specifying explicitly which cubie is in a given cubicle, the O-view notices the orientation in three space of whichever cubie is in the cubicle. The 24 possible values of the orientation are just the symmetries of a solid cube, which is called the cube group (not the Rubik's Cube group, a Platonic solid type cube). 3x3 rotation matrices may be used to write down the 24 elements of the cube group (these matrices are listed in A3).

Each position will be represented by a function whose domain is the set of cubicles, and whose range is the 24 element cube group. The Cube has now become a large set of matrix valued functions of vectors. Structures like this are usually called tensor fields, and are found throughout physics. The electromagnetic stress tensor is one, for example.

The S-view representation made a distinction between positions and operators on positions; this O-view does not. The 6N single moves are just the same as the 6N positions closest to the solved position. Every position can be thought of as an operator. Given two positions A and B, the position resulting from doing first A and then B is

$$(11) \qquad C(\vec{v}) = B(\vec{v}) \cdot A\left[B'(\vec{v}) \cdot \vec{v}\right]$$

This needs a bit of explaining. "$\vec{v}$" is a column vector representing a particular cubie. B' is, as always, the inverse of B. $B'(\vec{v})*\vec{v}$, the product of a 3x3 matrix and a 3x1 column vector, is a vector representing the cubicle which the cubie now in v was in before B was done. Therefore, the entire formula is the product of two rotation matrices describing the orientation of the particular cubie that ends up at $\vec{v}$.

While this may seem amazingly roundabout and complicated, it is actually reasonably elegant. Every cubie is treated in exactly the same way, and the location of each is readily available. A position which cannot be reached by legal moves can be described as easily as a legal position. And, nowhere is it necessary to create matrices of indeterminate size.

Further work on this representation is outside the scope of this paper, but an approach along these lines might yield some interesting results.

V. Conclusions

Around the turn of the century, a man named Albert Einstein developed a theory describing the interaction of time and space and the speed of light. His understanding of these concepts was particularly elegant and simple because he was able to think inside an unorthodox but peculiarly appropriate framework of assumptions.

The Cube may not be as profound as relativity, but trying to figure it out is a lot of fun. Almost as much fun as learning physics, and for essentially the same reason: elegance. Elegance is the quality of being fundamentally simple, yet decidedly non-trivial. Most mathematicians and many physicists thrive on elegance.

The trick to figuring out any system is finding ways of thinking which make it intuitive. Different aspects of the system may even require different viewpoints. Mechanics, for instance, is usually done within Newton's frame of reference when treating speeds much less than c, and according to Einstein when the speed is near c. In the same way, the S-view an the O-view treat different regimes of the Cube.

Inside or even defining a point of view, invariants and models are important tools in developing an understanding. This is hardly news; people have used them for centuries. Nevertheless, these techniques are so fundamental that an illustration can be helpful. The purpose of this paper has been to present the Cube as such an illustration.

The big question is how these viewpoints and what-not can be found. Answer: inspiration. Searching for general principles helps, too. Most of the laws in the "Theory of the NxNxN O-view Cube" were found empirically, and only later understood within a formal system. The procedure cannot be defined more precisely than this; too much of creating science is art.

But one guiding principle in the quest for comprehension stands out. Strive for elegance. A theory that is too messy is bound to be wrong. The facts are important too, but by themselves do not impart wisdom. Understanding means finding a nice, simple theory that explains what is going on. It means elegance.

Simplicity and elegance.

Glossary

adjoint:	If A and B are operators, ABA' is an adjoint of B. (A' is A-inverse.)
algorithm:	In this paper, a sequence of moves which acclomplishes some well defined task.
class:	Two cubie types belong to the same class if they are essentially the same. See fig. 8 for the basic classes.
commop:	An operator made from adjoints and commutators. See appendix A2.
corner:	A class of cubies. See fig. 8.
Cube, the:	What this paper is about. An array of smaller cubies which can be moved about by rotating planes of the Cube. The 3x3x3 version is sold in stores under the name "Rubik`s Cube" or the "Magic Cube."
cube group:	A structure of group theory. The group has 24 elements which represent the symmetries of a solid cube. See any good group theory text
cubicle:	The place in space where a cubie lives.
cubie:	One of the smaller cubes that make up the Cube (see fig.1). On the 3x3x3 they are labeled by their exposed colors.
edge:	On the 3x3x3, the type of cubie with two

	faces showing. Otherwise, the normal edge of any cube.
elegance:	The quality of being fundamentally simple yet decidedly non-trivial. See "God."
face:	The outside flat part of a cube.
God:	"The simplest of all." (see the dedication)
God`s Algorithm:	The shortest sequence of moves between any two positions.
group theory:	A set of mathematical structures satisfying a certain group of restrictions (ha ha). Check your local math text for more detail.
half 8 face:	A class of cubies. See fig. 8.
invariant:	Any variable which remains unchanged under transformations of the system.
kluge:	The quality of being decidedly inelegant.
layer:	The cubies showing on the outside of an NxNxN Cube are the N-layer.
location:	A place in space. See "cubicle."
M(N):	A lower bound on the size of the Cube network. See equation (8).
move:	A quarter rotation of any plane of the Cube. In the S-view, the center planes do not move.

network: In this paper, the topological space created by assigning a point in space to every possible position of the Cube, and connecting points separated by a single move by a line.

O-view: Pictures the Cube as N cubed cubies, each with one of 24 orientations. All moves and positions are defined relative to an absolute space.

operator: A term borrowed from group theory. Here it refers to a sequence of moves.

orientations: In the S-view, the ways that a cubie can fit into a given cubicle. In the O-view, an element of the cube symmetry group (see A3).

parity: Describes whether an even or odd number of moves have been done to a given layer.

physics: The attempt to understand "real world" systems, i.e. the Universe, the sub-atomic world, and what lies in between.

position: A state of the Cube. Defined differently for S- and O-views. See the Positions chapter

possible position: A position that can be reached by a sequence of legal moves, starting at the solved position.

representation:	A mathematical structure that has the same properties as the system in question.
S-view:	A way of looking at the Cube which only notices the colors on the outside faces. It is especially used in the context of the edges and corners of the 3x3x3 Cube.
single edge:	A class of cubies. See fig. 8.
slice:	A plane of the Cube.
solution:	A string of moves which will take the Cube from its current position to the solved position.
T(N):	The total number of possible positions for an NxNxN Cube (O-view).
thesis:	Defined by example: this paper is a thesis. (Who says self-reference doesn't work?)
turniness:	An invariant defined on the corners of any layer.
type:	Two cubies are of the same type if they can be moved into each other's cubicles.
understanding:	To have some notion of the what and the why.

A1. the Theory of the NxNxN 0-view Cube

First Law: Cubies may be grouped into different types which interact through parity consistency.

Second Law: Flippiness and turniness are conserved.

Third Law: Every cubie has exactly 24 orientations.

Fourth Law: Every orientation corresponds to a unique location.

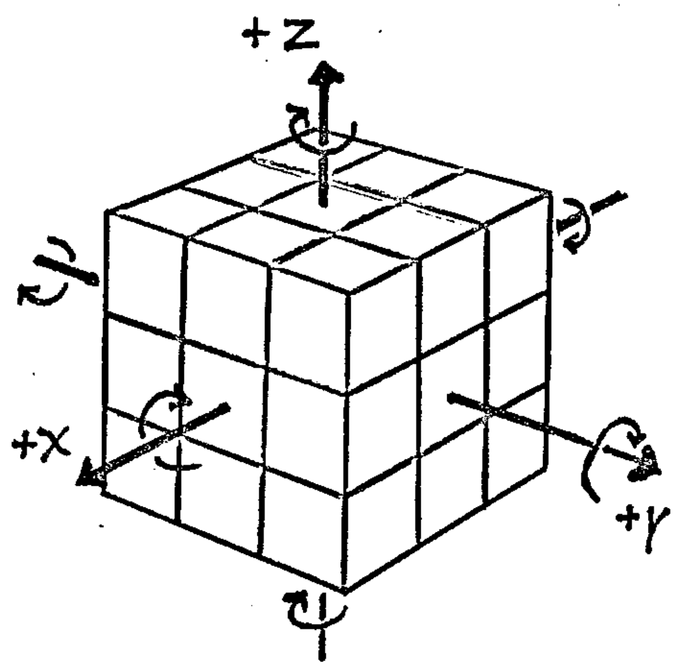
Fifth Law: Commutators and adjoints are enough to manipulate any given type without disturbing the rest of the Cube.

Sixth Law: The size of the space of all possible positions is proportional to the number of cubies.

A2. sample commops

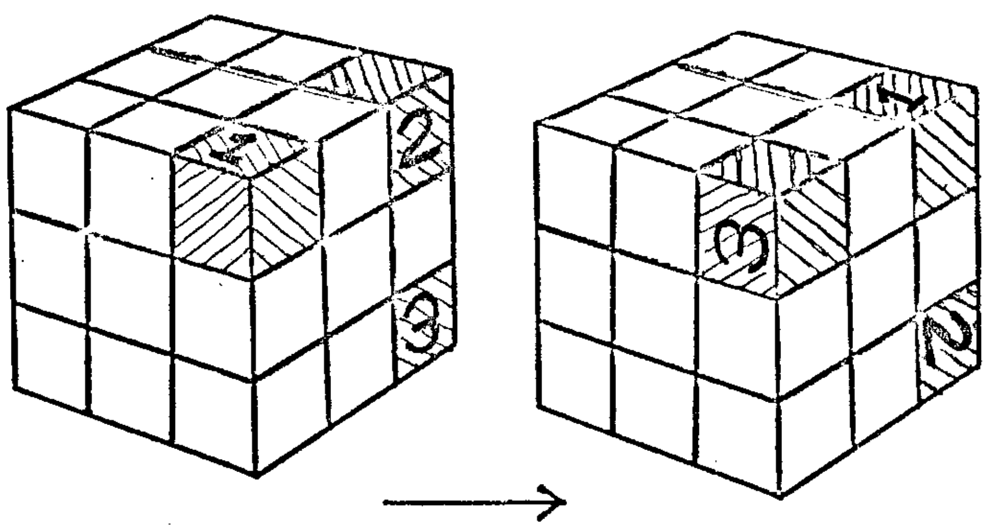
Notation: The positive rotation direction is defined as shown. [3x] stands for a quarter turn of the plane x=3 (each cubie has size one). Periods are used to seperate moves. [1y.1y] may be shortened to [y"]. The inverse of [-z] is [-z']. Only the cubies marked are affected.

Note that the central cubie always stays in the same cubicle, and may be oriented by the 0x, 0y, and 0z moves.

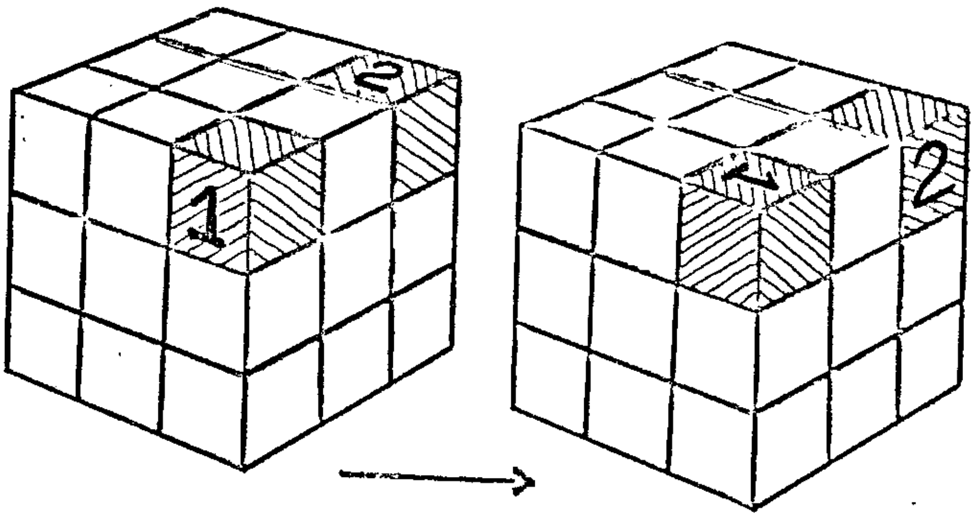


corner

(x.-z.x').z.(x.-z'.x').z'

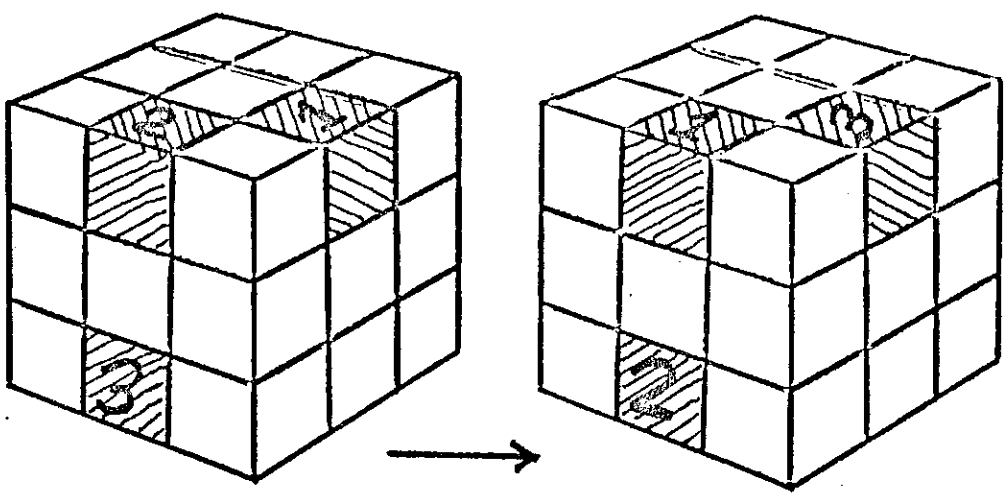


$$(x.-z''.x').(y'.-z''.y).z.(y'.-z''.y).(x.-z''.x').z'$$

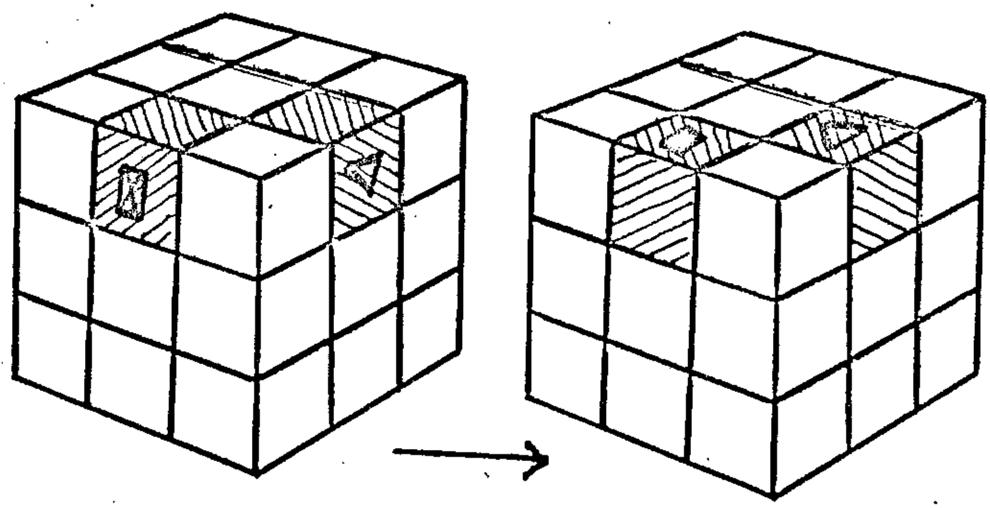


single edge

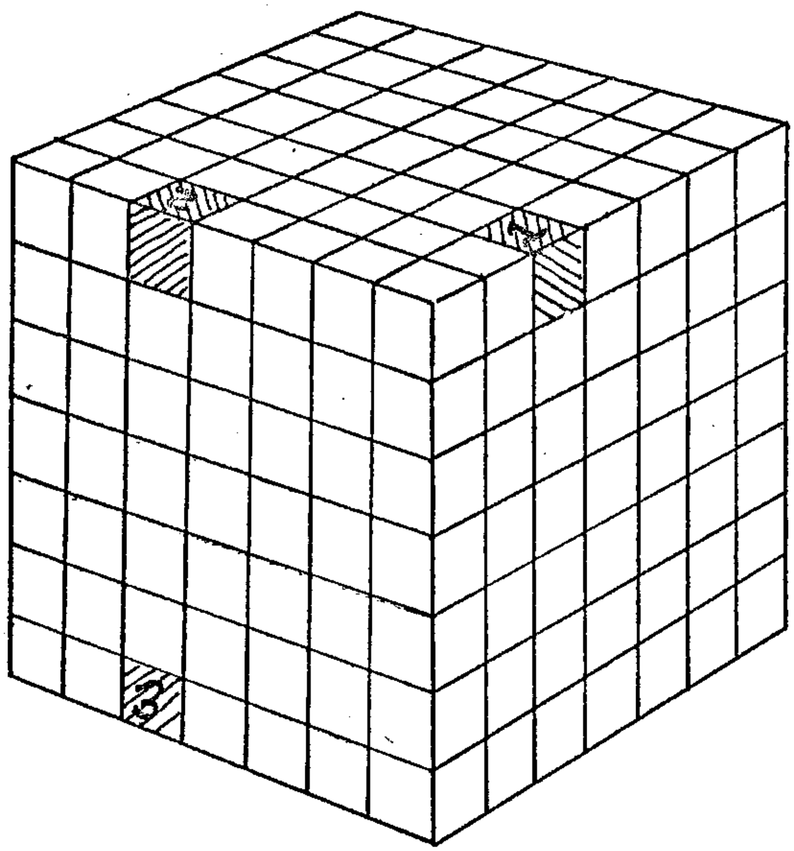
(0x.-1z'.0x').1z.(0x.-1z.0x').1z'



(y'.0z.y".0z".y').z'.(y.0z".y".0z'.y).z



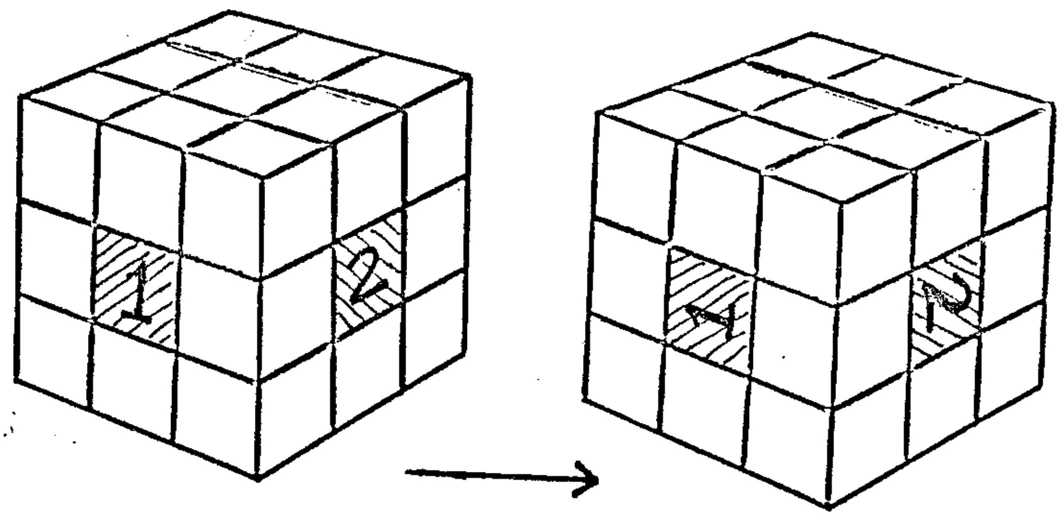
double edge



1→2→3→1

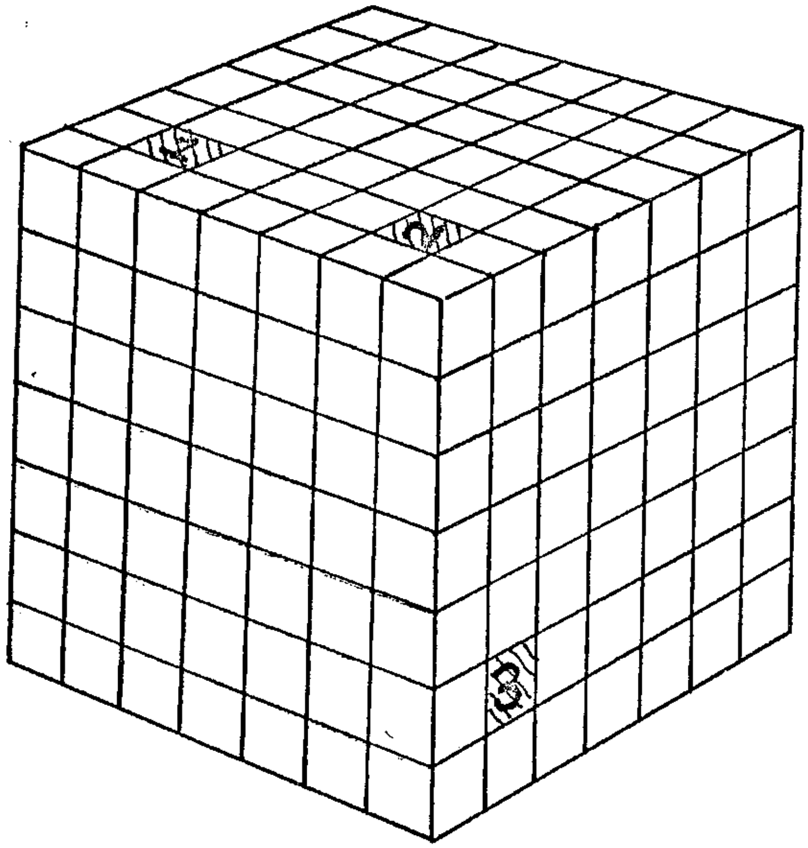
Use the same sequence shown for the single edges.

center face



(y.-y.x'.-y'.y').0z.(y.-y.x.-y'.y').0z'

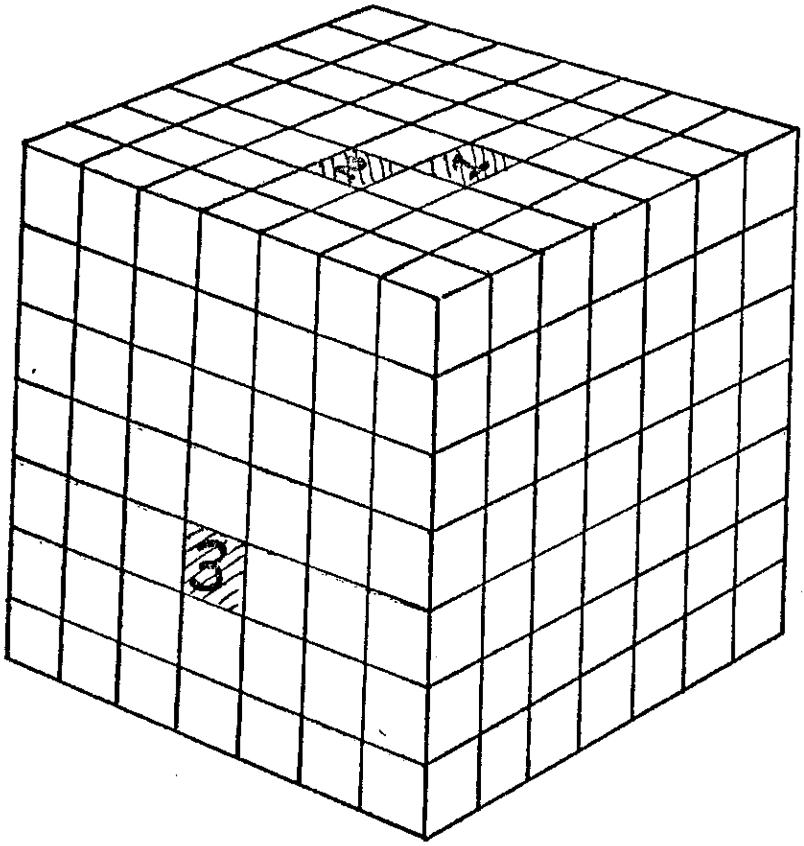
corner face



$1 \rightarrow 2 \rightarrow 3 \rightarrow 1$

(-2y'.-2z'.-2y).3z.(-2y'.-2z.-2y).3z'

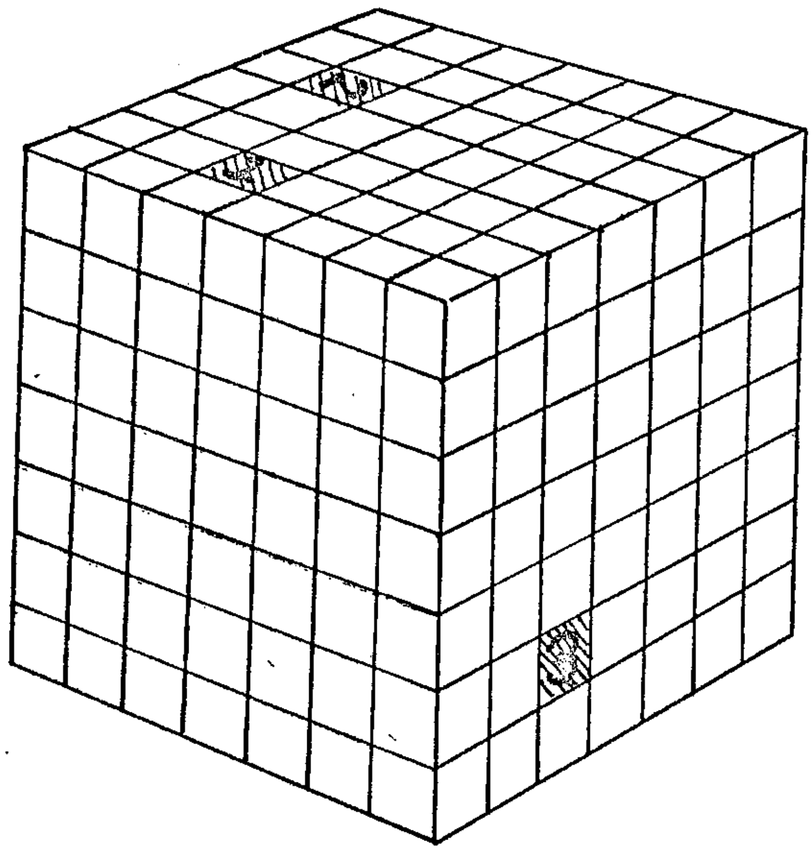
edge face



$1 \rightarrow 2 \rightarrow 3 \rightarrow 1$

(0x.-1z'.0x').3z'.(0x.-1z.0x').3z

half 8 face



1→2→3→1

(-1y'.-2z.-1y).3z'.(-1y'.-2z'.-1y).3z

A3. the 24 element cube group

These matrices represent the 24 possible orientations of a solid cube in 3-D space. Each is a rotation about one of the three symmetry axes, corner, edge center, or face center. Any vector pointing along the rotation axis of a particular operator is an eigen-vector of that operator. The determination of which axis corresponds to which matrix has been left as an exercise for the reader.

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$$

$$\begin{matrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{matrix}$$

$$\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}$$

$$\begin{matrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{matrix} \qquad \begin{matrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{matrix}$$

$$\begin{matrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{matrix}$$

$$\begin{matrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{matrix}$$

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