Chapter 2

Patterns and Proof

> In mathematics, if a pattern occurs, we can go on to ask, Why does it occur? What does it signify? And we can find answers to these questions. In fact, for every pattern that appears, a mathematician feels he ought to know why it appears.
>
> **W. W. Sawyer** (; -)

2.1 Patterns, Impostors and the Limits of Inductive Reasoning

This is a book about patterns. As several earlier quotations suggest, mathematics is very much the science of patterns. Patterns of various forms play essential roles in all of the books in this series, arising in art, dance, music, geometry, games, puzzles, reasoning, etc. While the Introduction suggested that there is not one universal definition of a pattern - we hope that you are becoming more adept at noticing patterns all around you.

And we hope that you are becoming more curious about how and why these patterns work; how their formation is born; the morphogenesis of these patterns.

2.1.1 Regions in a Circle

Figure 2.1 shows the first three stages in a pattern that arises from geometry - a pattern we will call ***Regions in a Circle***.

1. What pattern do you notice that determines what points and lines are added as we move from one stage of the pattern to the next?

2. Use this pattern to draw the next stage, i.e. the fourth stage, in this pattern.

3. Continue this pattern to draw the fifth stage in this pattern.

4. Into how many regions has the circle been decomposed in the first stage?

5. In the second stage? In the third stage?

6. Using the additional stages you have drawn, complete the following table:

Stage	Regions
1	2
2	
3	
4	

7. Do you see a clear pattern formed by the number of regions? Describe it precisely.

8. Use this pattern to predict the number of regions that are formed in the fifth stage in this pattern.

9. Now draw the fifth stage in this pattern and carefully count the number of regions that are formed.

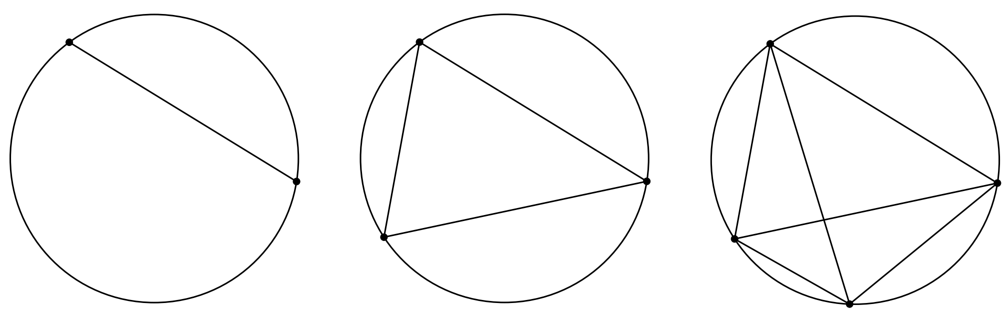

Figure 2.1: Stages 1, 2, and 3 of the Regions in a Circle pattern.

The pattern that you found in Investigation **7** is a beautiful, simple, and important numerical pattern, an examples of *exponential growth*. Unfortunately, as you have seen, this is pattern does *not* match the number of regions in a circle. The pattern fit the data for through the first five stages, but then failed. It is an impostor.

Why did this happen? It happened because human beings naturally use ***inductive reasoning*** - they draw general conclusions from limited evidence. We do this all of the time in our everyday experiences. We also do this in science when we apply the scientific method. If the consequences of our hypotheses fit observed data well enough in repeatable scientific experiments they become known as scientific theories.

2.1.2 Partitions

Let us consider another pattern. *Prime numbers* are important because they are the building blocks of all positive integers via multiplication. The number 42 can be uniquely *factored into primes* as $42 = 2 \times 3 \times 7$. But what if we want to build positive integers using addition? Well, $1 = 1$. And $2 = 2$ but also $2 = 1 + 1$. When we write a positive integer as the sum of positive integers these decompositions are called **partitions**. So the number 1 has only one partition while the number 2 has two partitions.

10. Find all of the different partitions of the number 3. (Note: we do not consider $2 + 1$ and $1 + 2$ to be different since they use exactly the same numbers.)

11. Use your previous answer to complete the following table:

Integer, n	Number of Partitions of n
1	1
2	2
3	

12. How many partitions do you think the number 4 will have?

You are making an educated guess in Investigation **12** based on evidence that you have collected. In science this is generally called a hypothesis. In mathematics the word typically used is ***conjecture***.

13. Find all of the partitions of 4.

14. Is you conjecture in Investigation **12** correct?

15. Use your previous answers to complete the following table:

Integer, n	Number of Partitions of n
1	1
2	2
3	
4	

16. How many partitions do you think the number 5 will have?

17. Find all of the partitions of 5.

18. Is your conjecture in Investigation **16** correct?

19. Use your previous answers to complete the following table:

Integer, n	Number of Partitions of n
1	1
2	2
3	
4	
5	

20. How many partitions do you think the number 6 will have?

21. Find all of the partitions of 6.

22. Is your conjecture in Investigation **20** correct?

23. Use your previous answers to complete the following table:

Integer, n	Number of Partitions of n
1	1
2	2
3	
4	
5	
6	

24. How many partitions do you think the number 7 will have?

25. Find all of the partitions of 7.

26. Is your conjecture in Investigation **24** correct?

27. Use your previous answers to complete the following table:

Integer, n	Number of Partitions of n
1	1
2	2
3	
4	
5	
6	
7	

28. How many partitions do you think the number 8 will have?

29. Find all of the partitions of 8.

30. Is your conjecture in Investigation **28** correct?

31. What do you think of inductive reasoning now? How well has it served you here?

Would you like to know the pattern? Here's an algebraic description - as the number n gets larger and larger, the number of partitions of n is better and better approximated by the function

$$\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}},$$

a result which was established by **G.H. Hardy** (English mathematician; 1877 - 1947) and **Srinivas Ramanujan** (Indian mathematician; 1887 - 1920) in 1918. The story behind the collaboration between these mathematicians is great and tragic. The self-taught Ramanujan was miraculously discovered by Hardy but then died at age 32 shortly after having been diagnosed with tuberculosis likely caused by his move from India to Cambridge to study with Hardy. Ramanujan's amazing discoveries of beautiful patterns of *partition congruences* has seen great rejuvenation in recent years with spectacular, surprising breakthroughs.[1]

Back to the pattern. Wow, is that formula complicated enough for you?

[1] For more on these beautiful and important patterns see Discovering the Art of Mathematics - Number Theory.