Chapter 5

Astrophysical Applications

The last two chapters have explored two rather different topics: (i) Newton's discovery of the law of universal gravitation, according to which massive particles exert an inverse-square-law gravitational force on one another and (ii) the rotational dynamics of extended (and in particular rigid) bodies. Our goal in this present chapter is to develop some applications of and connections between these ideas, by surveying a number of interesting discoveries from the period between Newton and today in which gravitation and/or rotation play some interesting role. We focus on applications from astronomy and astrophysics, considering (in proper Newtonian spirit) the Earth as part of the heavens. Especially the latter parts of this chapter depart somewhat from the earlier practice of explaining always not just "what is true" but also "how it was figured out." We instead just survey some interesting conclusions that have emerged from more recent research, without giving all of the historical background that would make every detail clear. One goal is to sketch some of the ways the two foundational topics of the previous chapters play an important role in these more recent discoveries. Another goal is simply to tempt you to want to learn more about these things in more advanced coursework.

5.1 The Shape of the Earth

As discussed in Chapter 1, the Ancient Greeks knew, some 2000 years prior to Newton's theory of gravitation, that the Earth was a sphere. And it is – *approximately.* Of course the surface of the Earth is marked with hills, valleys, and mountains. Such features are produced by terrestrial causes such as volcanoes and erosion; they are not the departures from perfect sphericality that will concern us here, as they have nothing (or at any rate, less) to do with rotation and gravitation.

Instead we will focus on an interesting systematic departure of the Earth from perfect sphericality: it "bulges" slightly at the Equator and is in fact a slighly *oblate spheroid.* The oblateness can be quantified this way: the "radius" of the Earth at the poles is a bit less than the "radius" at the equator. The difference is small compared to the radius itself, but surprisingly big on human scales. It is about 21.3 kilometers, or about 13 miles, or about 0.335% of the Earth's average radius:

$$f = \frac{R_E - R_P}{R} = .00335. \tag{5.1}$$

The earliest observational evidence pertaining to the Earth's oblateness was acquired in the 1600's and was noted, by Newton, in the Principia:

> "some astronomers, sent to distant regions to make astronomical observations, have observed that their pendulum clocks went more slowly near the equator than in our regions. And indeed M. Richer first observed this in the year 1672 on the island of Cayenne. For while he was observing the transit of the fixed stars across the merdian in the month of August, he found that his clock was going more slowly than in its proper proportion to the mean motion of the sun, the difference being [2 minutes and 28 seconds] every day. Then by constructing a simple pendulum that would oscillate in seconds as measured by the best clock, he noted the length of the simple pendulum, and he did this frequently, every week for ten months. Then, when he had returned to France, he compared the length of this pendulum with the length of a seconds pendulum at Paris (which was 3 Paris feet and 8 3/5 lines long) and found that it was shorter than the Paris pendulum, the difference being 1 1/4 lines."

The idea here is that the period of a pendulum depends on its length and the local acceleration of gravity, g, according to

$$T = 2\pi\sqrt{\frac{L}{g}}. \tag{5.2}$$

Astronomers had constructed very precise pendulum clocks, whose lengths were carefully "tuned" to tick precisely once per second. What was found, however, was that such clocks failed to keep accurate time if they were transported too far to the north or south, i.e., to a different latitude. The obvious explanation for this would be that different weather (e.g., changes in temperature or humidity) caused the *length* of a given pendulum to change a little bit when it was transported to a new latitude. But even when such effects were corrected for, the inconsistency persisted. So the only possible conclusion was that the Earth's gravitational acceleration, g, was not actually a constant – as it would have to be for a spherically symmetric Earth – but instead varied slightly with latitude.

As Newton reports, in order to tick with the same period, a pendulum at the Equator must be a little *shorter* than one in "our regions." It is clear from the above formula for the period that this implies that g is a little *smaller* near the Equator than it is closer to the Poles. This can be understood as the result of two related factors: the rotation of the Earth, and the Equatorial bulge which is caused by the rotation.

To begin with, think of the Earth as a perfect sphere with an additional layer of matter piled up near the Equator, as in Figure 5.1. An observer at point C in the Figure is a distance h *farther away* from the dominant, spherical part of the Earth's mass, than

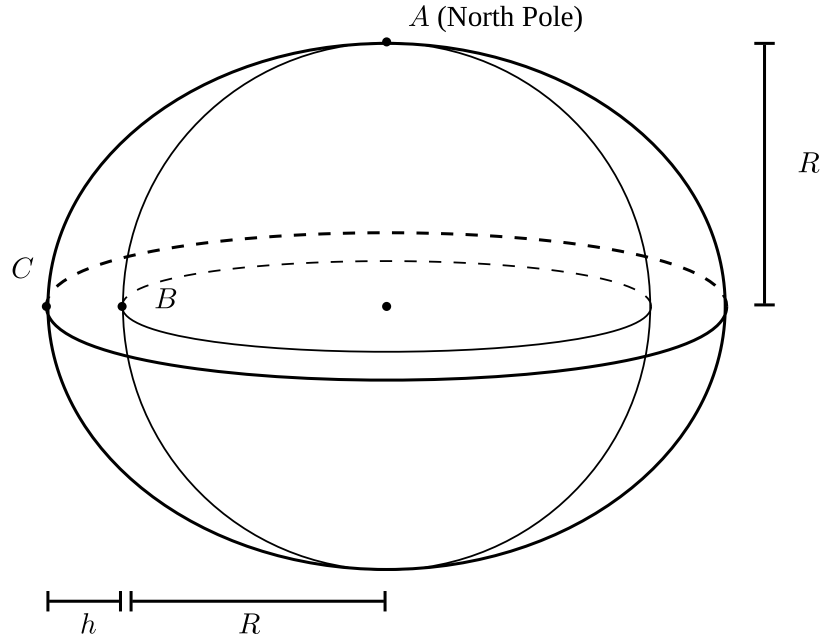

Figure 5.1: The Earth's Equatorial bulge: the Earth can be thought of as a perfect sphere of radius R, plus an additional layer of matter, thickest around the Equator, where its thickness (i.e., the difference between the "radius" at the Equator and the "radius" at the Pole) is h.

an observer at point A. This tends to decrease g near the Equator since the strength of the gravitational effect produced by the Earth's spherical core, falls off with distance. On the other hand, an observer at C has an extra layer of matter (of thickness h) right below him, and this tends to *increase* his g compared to an observer at A. It turns out (but is certainly not obvious) that the former effect is bigger in magnitude than the latter, i.e., the overall effect of the Equatorial bulge is to decrease g at the Equator relative to the Pole.

Actually, the fact that the Earth is *rotating* also contributes to the variation of g with latitude. This is because we usually define g as the acceleration that we would observe for a freely-falling object *from a reference frame that is attached to the Earth.* But since the Earth rotates, such a reference frame is not inertial – and so we cannot expect Newton's second law to apply! As we will discuss in detail shortly, we can still use Newton's laws in non-inertial reference frames if we introduce certain fudge factors called "inertial forces" – the most important and familiar of which is the so-called "centrifugal force" which tends to pull objects away from the axis of rotation. The magnitude of this (fictitious) centrifugal force turns out to be proportional to the mass m of the object it acts on, just like the gravitational force. So in practice the centrifugal force cannot be distinguished from a true gravitational force – which is why the two are usually lumped together and jointly described as an "effective gravitational force."

Of course, no such centrifugal force really exists. The point is, if one finds oneself in a non-inertial reference frame, it will *feel like* they do. And it is often convenient to indulge those feelings and use a non-inertial reference frame for the analysis of certain physical phenomena, even though, in principle, things could always be analyzed using an inertial reference frame (and only real forces!) too.

In any case, the immediate point is that, according to observers on the Earth, there is a centrifugal force which opposes and partly counteracts the gravitational force. The "effective" gravitational force on an object of mass m – which is equal to m times the effective gravitational accleration $\vec{g}_{eff}$, this being a definition of $\vec{g}_{eff}$ – is the vector sum of these two. Since the centrifugal force is strongest (and also most directly opposite the true gravitational force) for observers at the Equator, the rotation of the Earth also contributes to the systematic decrease of g_{eff} near the Equator.

Note also that these two causes of the systematic variation of g with latitude – the oblateness of the Earth and its rotation – are not unrelated. The earth is oblate *because* it rotates! It is precisely the centrifugal force which causes the (only semi-rigid) Earth to bulge out around its waist.

To briefly mention some of the interesting history: in the mid 1700's, scientists undertook a new, more direct method of measuring the shape of the Earth. They measured the actual distance, in miles, along the surface of the Earth that corresponded to moving North or South by one degree of Latitude, for different Latitudes. As expected on the basis of Newton's theory, the distance was a bit less near the Poles than near the Equator – i.e., the Earth really did bulge around the Equator. Today, the amount of oblateness or flattening can be measured very precisely from space using satellite images. And ground-based techniques of measuring g_{eff} are so precise that tiny local variations can be used to locate valuable underground deposits of natural resources!

Our goal in the rest of this section will be to understand, in a little more detail, how the rotation of the Earth, coupled with its self-gravitation, accounts for the observed amount of the Earth's oblateness. We'll then discuss the use of non-inertial (rotating) reference frames and the associated centrifugal forces, and apply these ideas to analyze again the relationship between the Earth's rotation rate, its oblateness, and the variation in the effective gravitational acceleration g_{eff} with latitude.

5.1.1 The Earth's Oblateness

If the Earth were a perfectly rigid sphere, and it were set rotating, nothing would happen. It would retain its spherical shape. But a somewhat elastic or liquid Earth will be flung outward, away from the axis of its rotation – just as pizza dough is stretched outward when it is tossed, spinning, into the air. Although this does not match the actual chronological process by which the Earth achieved its present shape, it is clarifying, in trying to derive a quantitative relationship between the Earth's rotation rate and its oblateness, to have in mind the following story: suppose the Earth used to be a perfectly rigid sphere rotating at a certain rate ω, but then "softened" and hence relaxed into its present oblate shape. For example, suppose it used to be a big, perfectly-spherical ice cube which then *melted*, allowing the water to flow into a new, energy minimizing, equilibrium configuration.

A crucial point is that the rearrangement of matter that occurs when the Earth "melts" will be produced exclusively by internal forces. Indeed, for the moment, we may simply ignore the fact that the Earth orbits the Sun, and instead imagine it to be rotating on its axis at some fixed point in otherwise-empty space. There simply aren't any relevant external forces at all, so clearly whatever rearrangement occurs must be the result of purely internal forces. And since, as shown in the previous Chapter, such internal forces will not produce any net torque, *the angular momentum of the Earth will have to remain constant even as it melts and adjusts its shape.*

Here's why this is so crucial. Suppose we ignored it and made the following argument. The rotating earth has some rotational kinetic energy

$$KE = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} I \omega^2 \tag{5.3}$$

and also some gravitational potential energy

$$PE \approx -\frac{3}{5}\frac{GM^2}{R}. \tag{5.4}$$

Now we contemplate the possibility that some of the matter from near one of the poles (point A in Figure 5.1) should move toward the Equator (which is clearly how we're going to get from a sphere to an oblate spheroid). Suppose it moves first along the (quarter-circle) path from point A to point B, and then along the straight line from B to C. During the first part of the path, it is always moving precisely horizontally, maintaining a fixed "altitude". So there is no change in its gravitational potential energy. But then, in moving from B to C, it has to move "uphill", which *increases* the overall gravitational potential energy. And since, by virtue of the rotation, the matter at the Equator has to be moving *faster* than the matter at the Pole, moving some matter from the Pole to the Equator entails an increase in the kinetic energy, too. And so, apparently, *the total energy must increase if any matter is moved from the Pole toward the Equator.* And so an initially-spherical Earth that "melts" certainly should *not* spontaneously acquire an oblate shape!

Of course, that argument is wrong, for the reason we've already hinted at. The problem is that it assumes that the Earth's overall rotation rate ω is the same before and after the contemplated movement of some matter from the Pole to the Equator. But, as we have argued, it isn't the angular velocity ω that would be constant if the initially spherical Earth melted and reconfigured itself – rather, its (spin) angular momentum $L = I\omega$ would be constant.

Moving some matter from the Pole (where $r_\perp = 0$) to the Equator (where $r_\perp = R$) of course *increases* the Earth's moment of inertia, I. So the conservation of angular momentum implies that the angular velocity ω must actually *decrease.* And since the kinetic energy is proportional to I to the first power, but ω to the *second* power, this means that the contemplated re-organization of matter will actually *decrease* the overall kinetic energy of the Earth: the kinetic energy of the one little blob of mass that moved will indeed increase, but the rest of the Earth will slightly slow its rotation and hence decrease its overall kinetic energy – resulting in a net decrease.

It turns out that, at least for a while, this decrease in the net kinetic energy is bigger than the associated increase in the potential energy. So matter will spontaneously "flow" from the region near the Poles to the region near the Equator. At some point, though, an equilibrium is reached, beyond which further transfer of material from the Pole to the Equator would decrease the kinetic energy less than it increased the potential energy – i.e., such further transfer of material would increase, rather than decrease, the total energy. We can find a quantitative expression for the equilibrium shape of the Earth by noting that, in equilibrium, the total energy change produced by moving a tiny piece of matter from the Pole to the Equator should *vanish*.

To proceed with the calculation note that the total kinetic energy can be written in terms of the spin angular momentum $L = I\omega$ as follows:

$$KE = \frac{1}{2}I\omega^2 = \frac{L^2}{2I}. \tag{5.5}$$

Now, by taking differentials on both sides, we can write the following simple expression for the change ΔKE in kinetic energy that is produced by a small change ΔI in the moment of inertia:

$$\Delta KE = -\frac{1}{2}\frac{L^2}{I^2}\Delta I = -\frac{1}{2}\omega^2\Delta I. \tag{5.6}$$

Note that for positive ΔI, the change in kinetic energy is *negative*. That's just what we reasoned out in words above. And note that, if we had forgotten about the conservation of angular momentum and just naively taken differentials of $KE = \frac{1}{2}I\omega^2$, we'd have gotten the same final expression but with the opposite sign: $\Delta KE = \frac{1}{2}\omega^2\Delta I$. And then we'd never be able to understand why the Earth bulges at its Equator.

Let's now try to develop an actual formula for the height of the bulge. Consider the particular sort of change contemplated above – a small chunk of matter, say of mass m, being moved from the Pole to the Equator. Since, as mentioned before, its $r_\perp$ (the quantity that enters into this chunk's contribution to the moment of inertia) increases from zero to R (the radius of the Earth), we have

$$\Delta I = mR^2 \tag{5.7}$$

and hence

$$\Delta KE = -\frac{1}{2}mR^2\omega^2. \tag{5.8}$$

Of course, technically speaking the R here should be the Earth's *equatorial* radius, not its polar radius. But we'll ignore this difference here, because it turns out not to make a significant difference. (Taking it into account would only introduce a small correction to the already-small thing we are here calculating: the difference between the equatorial and polar radii!)

Now what about the change in the gravitational potential energy associated with moving this chunk of matter from the Pole to the Equator? The idea is to first move the chunk "horizontally" along the (initially spherical) surface of the Earth, from A to B in the Figure. Since the Earth is spherical, the potential energy of a mass m hunk should be the same at B as it was at A, and so

$$\Delta PE_{A \to B} = 0. \tag{5.9}$$

We then have to move the hunk *up* a little bit, from point B to point C. Let's call this extra vertical distance h – it is just the difference between the Polar and Equatorial radii that we are trying to calculate. Then the gravitational potential energy change, when this one chunk of matter moves, is

$$\Delta PE_{B \to C} = mgh. \tag{5.10}$$

Of course, the relevant acceleration of gravity g will vary a little bit between B and C. But it only varies a little bit, and we can ignore this for the purposes of the present calculation.

We now need only plug Equations 5.8 and Equations 5.10 into the equilibrium condition

$$\Delta KE + \Delta PE = 0. \tag{5.11}$$

The result is

$$mgh - \frac{1}{2} m R^2 \omega^2 = 0 \tag{5.12}$$

or, solving for h and expressing the gravitational acceleration g in terms of Newton's constant and the physical properties of the Earth ($g = GM/R^2$),

$$h = \frac{R^4 \omega^2}{2GM}. \tag{5.13}$$

This is the amount by which the Equatorial radius of an approximately spherical body will exceed its Polar radius (assuming it's rigid enough to rotate as a whole, but also fluid enough to relax into this equilibrium configuration).

A nice dimensionless measure of the oblateness is the so-called "flattening parameter" f – the difference in the Equatorial and Polar radii, divided by the (say, average) radius

$$f = \frac{h}{R} = \frac{R^3 \omega^2}{2GM}. \tag{5.14}$$

What does this formula predict for the oblateness of the Earth? It is easy enough to plug in numbers: $R = 6.37 \times 10^6 m$, $M = 5.97 \times 10^{24} kg$, $\omega = 2\pi\,\text{radians/day} = 7.27 \times 10^{-5}\,\text{rad/sec}$, and $G = 6.67 \times 10^{-11} m^3/kg\, s^2$. The resulting prediction is

$$h = 11\, km \tag{5.15}$$

or

$$f = 0.0017 \tag{5.16}$$

which is about a factor of two shy of the actual observed numbers. As we'll see in the rest of this chapter, it's pretty good for these kinds of problems even to get the order of magnitude right. Often, and certainly here, there are a lot of really complicated details that we just ignore or approximate over. So getting within a factor of two definitely counts as achieving a decent quantitative understanding of the observed facts – and also leaves plenty of room for more sophisticated work in the future!

But actually here the factor-of-two discrepancy (between Equation 5.13 and the true value $h \approx 21.3$ kilometers) is a result of a pretty bad flaw in the above argument. (Did you notice it?!) We assumed that the potential energy of a hunk of matter at points A and B was the same, such that $\Delta PE_{A\to B} = 0$. That would indeed be true, as we said above, if the Earth were perfectly spherical. But of course the whole point of this discussion is that it *isn't*! And indeed, thinking about it qualitatively, it's pretty clear that in moving from point A to point B, we are moving *closer and closer* to the extra "belt" of matter surrounding the Equator – i.e., as far as *gravitation* is concerned, the path from A to B is going to be decidedly *downhill*. And so in fact $\Delta PE_{A\to B}$ will not be zero, but will be negative. It stands to reason that it, like $\Delta PE_{B\to C}$, should be roughly proportional to h – and indeed it turns out that these two contributions to ΔPE, one positive and the other negative, are of roughly the same order of magnitude:

$$\Delta PE_{A\to B} \approx -\frac{1}{2}\Delta PE_{B\to C} \approx -\frac{1}{2}mgh. \tag{5.17}$$

And so the *total* change in potential energy associated with the contemplated transfer of a small chunk of matter from the Pole to the Equator turns out to be more like

$$\Delta PE_{A\to C} = \Delta PE_{A\to B} + \Delta PE_{B\to C} \approx \frac{1}{2}mgh \tag{5.18}$$

which has the effect of *doubling* our earlier estimate for h, bringing the prediction much better in line with the actual observations. The rather subtle and difficult task of calculating $\Delta PE_{A\to B}$ will be further explored in the Projects.

Newton's theory of gravitation allows us to understand how primordial clouds of gas and dust could clump up under the mutual gravitational attraction of their parts, and form spherical blobs – the sphere being the natural result when lots of individual particles of matter try to get as close as they can to one another. The upshot of the above calculations is that Newton's theory allows us also to understand not only why the Earth and other heavenly bodies are more or less spherical, but also why and by how much they deviate from perfect sphericality due to rotation.

5.1.2 Rotating Reference Frames

We have just analyzed the oblateness of the Earth in terms of a certain trade-off in energies: if an initially-rigid and perfectly spherical rotating Earth were to melt, the gravitational potential energy would be increased by having some of the matter flow from the Poles to the Equator; but the overall kinetic energy would be decreased. Initially, the decrease would be greater than the increase, so matter *would* spontaneously flow toward the Equator – until an equilibrium is reached for which further such transfer of matter is energetically indifferent.

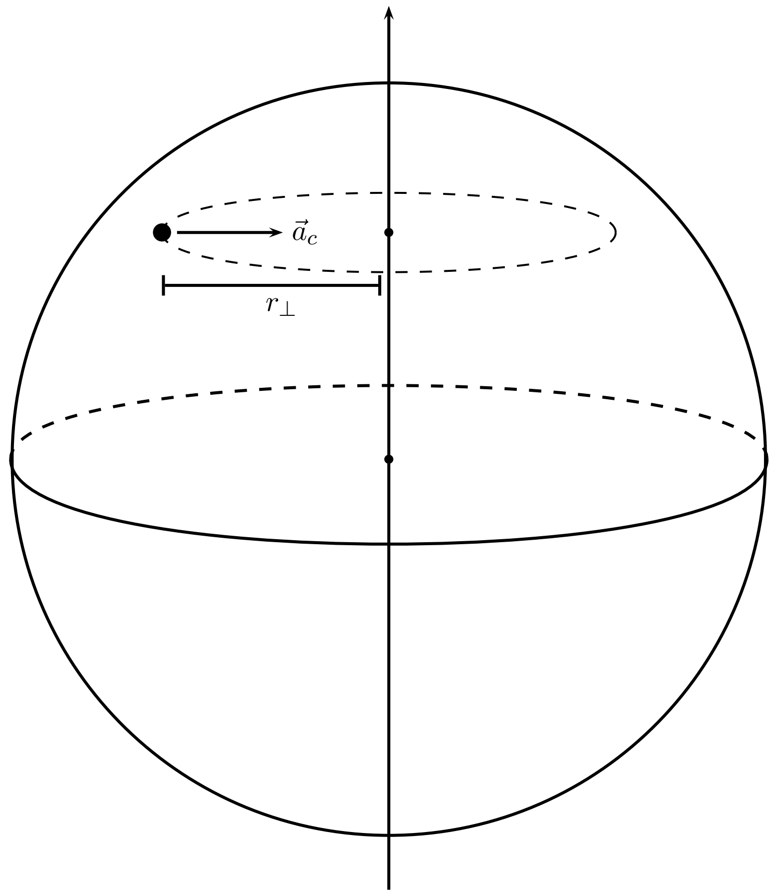

Figure 5.2: Some random particle in the Earth, undergoing uniform circular motion with radius $r_\perp$ and centripetal acceleration $\vec{a}_c$. The centripetal acceleration's magnitude will be $a_c = v^2/r_\perp = r_\perp \omega^2$, where ω is the Earth's angular velocity.

It is also possible to understand the oblateness by using a (non-inertial!) reference frame that co-rotates with the Earth. To see how to do this, let's first think about how Newton's second law, $\vec{F} = m\vec{a}$, applies to some particle of Earth if we use an inertial reference frame. Let's assume the particle is stationary with respect to the turning Earth, i.e., undergoing uniform circular motion with speed $v = r_\perp \omega$ and centripetal acceleration, directed perpindicularly in toward the Earth's rotation axis, of magnitude $a_c = v^2/r_\perp = r_\perp \omega^2$. See Figure 5.2.

The point is just that, according to Newton's second law, the net force acting on the object – the vector sum of whatever gravitational, frictional, normal, electric, magnetic, etc. forces are acting on it – will add up to its mass m times the centripetal acceleration $\vec{a}_c$. That is:

$$\vec{F}_{net} = m\vec{a}_c. \tag{5.19}$$

Since the centripetal acceleration is, well, toward the center, let us write this a little more explicitly as

$$\vec{F}_{net} = -m\omega^2 r_\perp \hat{r} \tag{5.20}$$

where $\hat{r}$ is a unit vector pointing radially outward (in the cylindrical rather than the spherical sense), i.e., perpindicularly outward from the central rotation axis.

Now, what if we contemplate the motion of this same particle from the point of view of a coordinate system that rotates around with the rotating Earth? The main point is just this: relative to such a coordinate system, the particle isn't ever moving! And so, in particular, its acceleration is zero. Since the question of what forces act is not in any way dependent on our (subjective, arbitrary) choice of reference frame, note that this makes Newton's second law *false*. The net force is *not zero*. Yet, as reckoned in this co-rotating reference frame, the acceleration *is zero*.

None of that should be too interesting or surprising, but is maybe clarifying about why the concept of inertial reference frames is so important for Newtonian dynamics (in particular, why the first law of motion is more than a mere special case of the second). What's interesting and surprising is that we can *make* Newton's second law hold, even in the non-inertial frame, by cooking the books a little bit.

Here is the trick: whatever reference frame we choose to use, Equation 5.20 remains true. How can we reconcile this with the fact that, in the co-rotating frame, the acceleration is zero? We may simply rewrite Equation 5.20 this way:

$$\vec{F}_{net} + m\omega^2 r_\perp \hat{r} = 0 \tag{5.21}$$

and interpret the right hand side as the mass m times the acceleration $\vec{a} = 0$ in this non-inertial frame! We can then interpret the left hand side as some kind of modified or "effective" net force: it is the sum of all the *real* forces and a fictitious *centrifugal force* of magnitude $m\omega^2 r_\perp$.

Let's try to come to grips with this by considering the simplest possible example: a rock sitting on the ground somewhere at the Equator. Suppose there are just two forces acting on the rock: a weight force of magnitude W and a normal force of magnitude N. (Of course, the weight force is down, toward the center of the Earth, and the normal force is up, away from the center of the Earth.) Since the rock is rotating around with the Earth it has centripetal acceleration of magnitude

$$a_c = \omega^2 R \tag{5.22}$$

where R is the radius of the Earth. So evidently it must be that the weight force is just a little larger in magnitude than the normal force: $W > N$. In particular, we must have that

$$W - N = m\omega^2 R \tag{5.23}$$

in accordance with Newton's second law.

Now what if we consider this same situation using a non-inertial reference frame that co-rotates with the Earth? It may seem at first that there is a contradiction: The weight force is bigger than the normal force, yet the rock doesn't accelerate! Ah, but there is also the centrifugal force which, despite not really existing, must be treated as real *if* *we insist on using this non-inertial frame of reference.* And then, of course, there is no problem: the weight force pulls the rock in one direction with a certain force, and the normal force and the centrifugal force *together* pull the rock equally hard in the opposite direction, resulting in zero acceleration:

$$W - N - m\omega^2 R = 0. \tag{5.24}$$

Note also that the centrifugal force is proportional to the mass m of the rock, just like the weight force $W = mg$. So it is conventional to group these two forces (one real, one fictitious) together into a single so-called "effective" gravitational force:

$$W_{eff} = W - m\omega^2 R = m(g - \omega^2 R) \tag{5.25}$$

where the quantity in parentheses is then defined as the "effective gravitational acceleration":

$$g_{eff} = g - \omega^2 R. \tag{5.26}$$

Of course, in the general case (of an object not necessarily at the Equator) we'd have to recognize the vector character of all these quantities. So the general formula for the effective gravitational acceleration is:

$$\vec{g}_{eff} = \vec{g} + \omega^2 r_\perp \hat{r}. \tag{5.27}$$

Now let's see how this relates to the oblateness of the Earth. Consider some random hunk of (say) water at the surface of the Earth at some latitude ϕ. See Figure 5.3. The important point is that the effective gravitational accleration (which determines the local meaning of "up" and "down" in the rotating coordinate system) will be tilted slightly away from its expected direction of "true down", i.e., toward the center of the Earth.

Since the surface of the Earth is largely liquid (and even the solid parts are relatively plastic on long, geological time-scales), its surface will everywhere be approximately perpindicular to the local $\vec{g}_{eff}$. And so if we can just calculate how $\vec{g}_{eff}$ varies with latitude, we can determine exactly the angle that "effective up" makes with "true up" at different latitudes, and from that understand the shape of the Earth.

Let's begin by breaking the centrifugal force up into "true horizontal" and "true vertical" components. The horizontal piece is

$$F_c^{horiz} = F_c \sin(\phi) = m\omega^2 R \cos(\phi)\sin(\phi). \tag{5.28}$$

The "true vertical" component is

$$F_c^{vert} = F_c \cos(\phi) = m\omega^2 R \cos^2(\phi). \tag{5.29}$$

Assuming that the true gravitational acceleration $\vec{g}$ is directed toward the center of the Earth, we then have that

$$g_{eff}^{vert} = g - \omega^2 R \cos^2(\phi) \tag{5.30}$$

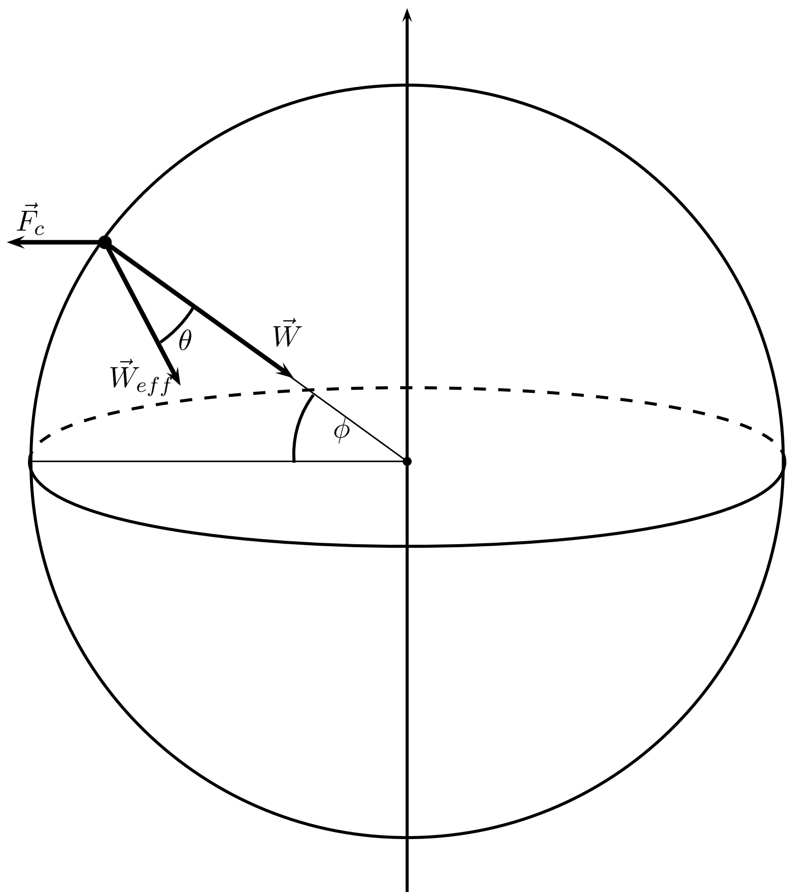

Figure 5.3: The gravitational ($\vec{W}$) and centrifugal ($\vec{F_c}$) forces acting on a hunk of, say, water at the surface of the ocean at latitude angle ϕ. The centrifugal force has magnitude $F_c = m\omega^2 r_\perp = m\omega^2 R|\cos(\phi)|$. The vector sum of these two forces (one real and one fictitious!), the effective gravitational force $\vec{W}_{eff}$, is also shown. The important point is that, because of the centrifugal force's contribution, $\vec{W}_{eff}$ does not point toward the center of the Earth, i.e., does not point in the direction we've been calling "true down." In equilibrium, the water's surface will be perpindicular to $\vec{W}_{eff}$, and so the surface will not be a perfect sphere.

which expresses the (small) reduction in g_{eff} with latitude that was primarily responsible for the effects noted first by Richer and discussed by Newton.

The horizontal component of $\vec{g}_{eff}$ will then be just the relevant component of the centrifugal force (divided by m):

$$g_{eff}^{horiz} = \omega^2 R \cos(\phi)\sin(\phi). \tag{5.31}$$

The angle θ that $\vec{g}_{eff}$ makes with "true vertical" at latitude ϕ is therefore given by $\tan(\theta) = g_{eff}^{horiz}/g_{eff}^{vert}$. But since the right hand side is very small, we might as well use the small angle approximation: $\tan(\theta) \approx \theta$. Moreover, since the second term in Equation 5.30 is small compared to the first term, we can here get away with approximating the angle as

$$\theta = \frac{\omega^2 R \cos(\phi)\sin(\phi)}{g} = \frac{\omega^2 R^3 \cos(\phi)\sin(\phi)}{GM} \tag{5.32}$$

where we have used the fact that $g = GM/R^2$.

Now imagine traveling along the surface of the Earth from the North Pole down to the Equator, and keeping track of the change in "true altitude" (distance from the center of the Earth) as one moves. A decrease in latitude by $d\phi$ corresponds to a linear distance $ds = R\,d\phi$ along a meridian of the Earth. Over this distance, the "true altitude" will increase by

$$dh = \theta\, ds = \frac{\omega^2 R^4 \cos(\phi)\sin(\phi)}{GM}\, d\phi. \tag{5.33}$$

And so the total increase in "true altitude" between the Pole and the Equator can be found by integrating:

$$h = \int dh = \frac{\omega^2 R^4}{GM} \int_0^{\pi/2} \cos(\phi)\sin(\phi) d\phi = \frac{\omega^2 R^4}{2GM} \tag{5.34}$$

which is the same result we got before in a different way. Or more precisely: this is the same *wrong* result we got before in a different way. And the reason we got the wrong result again is that we let the same wrong assumption creep in here! Before we wrongly, at first, assumed that there was no potential energy change associated with moving a hunk of matter from point A to point B of Figure 5.1. That is equivalent to assuming that the gravitational force does no work on a particle moving along the (quarter circle) path from A to B, which would be true precisely if $\vec{g}$ had no "horizontal" component, which is what we assumed here.

Of course, what we eventually realized before – that the journey from A to B is gravitationally "downhill" – implies here that $\vec{g}$ *does* have a "horizontal" component. Why? Because there is this extra belt of matter around the Equator which attracts our test particle and tilts the true gravitational acceleration g a little bit toward the Equator. And that means we *underestimated* the amount by which the surface of the Earth at latitude ϕ tilts relative to "true horizontal". Evidently this extra tilt that results from the not-quite-radial character of $\vec{g}$ contributes approximately as much to h as the (direct) centrifugal force contribution we already calculated.

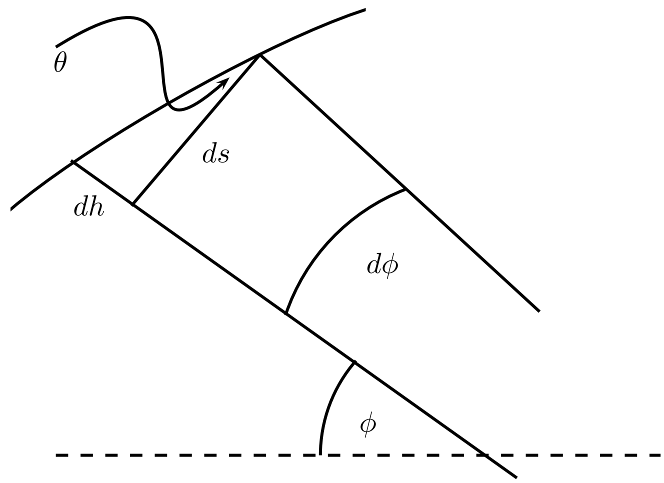

Figure 5.4: The surface of a bit of water at latitude ϕ. Over a small horizontal distance ds which spans a small latitude range $d\phi$, the height of the water increases (from the Polar to the Equatorial side) by an amount $dh = \theta\, ds$.

Let's finally return to the observations which began this whole discussion – the fact that the period of the same identical fixed-length pendulum varies somewhat with latitude, indicating that the local effective acceleration of gravity, g_{eff}, also varies somewhat with latitude. We've already written down equations for the horizontal and vertical components of $\vec{g}_{eff}$. In principle, the magnitude g_{eff} can be found from the Pythagorean Theorem, but since the horizontal component (despite its important role in determining the shape of the Earth!) is always small compared to the vertical component of g, it's a very good approximation to take

$$|\vec{g}_{eff}| = g_{eff}^{vert} = g^{vert} - \omega^2 R \cos^2(\phi). \tag{5.35}$$

Between the poles and the Equator, $\cos^2(\phi)$ varies between zero and one. So the part of the difference in g_{eff} between the Poles and the Equator that is attributable directly to the Earth's rotation is just

$$\Delta g_{eff} = R\omega^2 = 0.034\,\mathrm{m/s^2} \tag{5.36}$$

with the g_{eff} at the Equator of course being this much *smaller* than at the Poles.

Actual empirical measurement reveals that g_{eff} varies by just a little more than this:

$$\Delta g_{eff} = 0.052\,\mathrm{m/s^2}. \tag{5.37}$$

The extra discrepancy is of course due to the fact that g^{vert} itself varies a little bit with latitude, it being, evidently, $0.018\,\mathrm{m/s^2}$ smaller at the Equator than at the Poles.

The main reason for this difference was mentioned earlier: at sea level at the Equator, one is further from the center of the Earth, by a height h, than at the Poles. If one were on a *ladder* of this height (rather than a several-miles-thick slab of solid, gravitating earth!) above a perfectly-spherical Earth, the difference in g at the two locations would be

$$\Delta g = \frac{GM}{R^2} - \frac{GM}{(R+h)^2}. \tag{5.38}$$

To simplify this, it is useful to write the second term as

$$\frac{GM}{R^2}\left(1+\frac{h}{R}\right)^{-2} \tag{5.39}$$

and then use the purely mathematical fact that

$$(1+x)^n \approx 1 + nx + \frac{1}{2}n(n-1)x^2 + \cdots \tag{5.40}$$

for small x. (This can be derived, for example, by Taylor expanding the left hand side about $x = 0$.)

Since h/R is small, we may use this approximation and keep only the first-order term. The result is that

$$\frac{GM}{(R+h)^2} \approx \frac{GM}{R^2} - \frac{2GMh}{R^3} \tag{5.41}$$

or

$$\Delta g \approx \frac{2GMh}{R^3} = 2g\frac{h}{R}. \tag{5.42}$$

Plugging in numbers (in particular, the true value for h) gives

$$\Delta g = 0.066\,\mathrm{m/s^2}. \tag{5.43}$$

Of course, that's not quite right, because a mass at the Equator is not on a ladder of height h above a spherical Earth, but is rather supported by an enormous slab of gravitating material. This turns out, not surprisingly, to increase g at the Equator more than it does at the Poles, i.e., to contribute negatively to what we've been calling Δg. Evidently this extra negative contribution is just what brings our previously-calculated $\Delta g = .066\,\mathrm{m/s^2}$ in line with the empirically correct $\Delta g = .018\,\mathrm{m/s^2}$.

Notice that we have again here skirted the question of how to actually *calculate* the contributions to $\vec{g}$ that arise from the gravitational effect of the Earth's Equatorial bulge. This is not in principle all that difficult to treat exactly, but requires some rather sophisticated math. We'll take it up in the Projects and, at least, work up some order-of-magnitude estimates to convince us that everything makes sense.

There is one last thing we'll need in the Projects. We mentioned above that the actual, empirically-measured difference between the gravitational acceleration g (not g_{eff}, but the genuine gravitational field g) at the Pole and Equator is $\Delta g = 0.018\,\mathrm{m/s^2}$, where we are talking about sea level at both locations. We also just calculated that climbing to the top of a height-h ladder near the surface of the Earth has the effect of reducing g by $0.066\,\mathrm{m/s^2}$. It is then straightforward to calculate that the Δg – *between points at the same "true altitude" at the Pole and Equator* – is going to be:

$$\Delta g = (0.018 - 0.066)\,\mathrm{m/s^2} = -0.048\,\mathrm{m/s^2}. \tag{5.44}$$

Just for clarity, what this means is that the (genuine) gravitational acceleration at sea level at the Equator is $0.048\,\mathrm{m/s^2}$ *greater* than the (genuine) gravitational acceleration at a point just *above* the Pole such that the two points are equidistant from the Earth's center. Qualitatively, it of course makes perfect sense that, for two points equidistant from the Earth's center, the gravitational acceleration would be stronger at the point that is nearer to the bulging part of the mass distribution.

5.2 Tides

There is another way in which the Earth's surface bulges away from perfect sphericality, familiar to anyone who has ever visited the ocean: the tides. Let's try to understand the physical origin of the tides, first qualitatively and then with some mathematical analysis.

First some basic qualitative facts about ocean tides. At least at most locations on the Earth, there are roughly two high and two low tides per day – "roughly" because, strictly speaking, the average time between two subsequent high tides is not precisely 12 hours, but rather about 12 hours and 25 minutes. This is just half of 24 hours and 50 minutes, which happens to be the amount of time it takes a given point on the Earth to rotate all the way around and arrive at the same place – not the same place with respect to the Sun (24 hours) or the stars (23 hours 56 minutes), but *the same place with respect to the Moon*. So that is the first and most obvious piece of evidence that the tides are controlled, somehow, by the Moon.

Actually, even this was a controversial claim for a surprisingly long period in history. Many commentators had speculated that the Moon is somehow or other controlling the tides, but nobody understood *how* and nobody was able to explain satisfactorily why there were *two* high tides per day. A naive explanation involving the Moon would have, say, the Moon pulling the Earth's water toward it a bit, causing an extra-high pile-up of water on the side of the Earth facing toward the Moon, and an extra-low deficit of water on the side of the Earth facing away from the Moon. Then, as the Earth rotated (all the way around every 24 hours 50 minutes!) underneath the moon, a given point on the Earth's surface would pass alternately through the high- and low-water regions, resulting in one high and one low-tide per day. It's a nice story, but, unfortunately, it is contradicted by the observations.

Galileo also came up with a speculative theory in which the twice-per-day rising and falling of the tide was explained (in some way that is a little obscure, and not too important because it is definitely wrong) by some sort of interaction between the two primary motions of the Earth: its daily rotation and its yearly orbit around the Sun.

The point is just to acknowledge that the tides are confusing and complicated. They were only first properly understood by Newton, using (what else?) his theory of universal gravitation.

The basic idea of Newton's gravitational explanation of the tides is this. Since (in accordance with Newton's third law) not only does the Earth exert an attractive gravitational force on the Moon, but also vice versa, the Earth itself undergoes uniform circular motion (centered on the Earth-Moon center of mass point) and is thus constantly accelerating toward the Moon. But the parts of the Earth that are closest to the Moon will experience – because the gravitational force decreases with distance – a *stronger than average* gravitational attraction toward the Moon, while parts of the Earth that are farthest from the Moon will experience – for the same reason – a *weaker than average* gravitational attraction toward the Moon.

The point is that – *relative to this average attraction toward the Moon* (as embodied, say, by the gravitational acceleration of the point at the Earth's geometrical center) – the stuff on the side of the Earth nearest the Moon will be attracted (just a little bit) toward the Moon, while stuff on the side of the Earth farthest from the Moon will be (just a little bit) *repelled, away from the Moon*. And so stuff – like the water in the oceans – that is more or less free to flow around and re-position itself will tend to pile up at these two opposite positions on the Earth's surface. And that, obviously, is where it'll be high tide. And so a typical point on the Earth's surface will pass through *both* high-tide regions per day.

This is illustrated in Figure 5.5. The normal arrows represent the strength of the gravitational force exerted on that part of the Earth by the Moon. As discussed above, the points closer to the Moon experience a greater than average attraction to the Moon and the point farthest from the Moon experiences a smaller than average attraction to the Moon. In addition, points like those at the top and bottom of the Earth in the Figure experience an attraction that is approximately the same magnitude as average, but tilted at a slight angle. The double arrows represent the *difference* between the actual attraction at a point, and the average attraction. The upshot is clear: relative to the average motion of the Earth as a whole, the surfaces on the top and bottom (of the Figure) are pushed in/down, while the surfaces on the sides are pushed out/up. The result is something like the elliptical shape (technically a prolate spheroid) indicated in the Figure – though the extent of the tidal bulges is significantly exaggerated there.

One should think of this ellipsoid as an equilibrium shape that the surface of the oceans would make if this were the only relevant effect. But of course, the Earth itself is spinning around once per day (or once every 24 hours 50 minutes relative to the Moon). So, as a kind of first approximation, one should think of the oceans as always making roughly this equilibrium ellipsoid, with the tidal bulges essentially fixed in space relative to the Moon – but with the solid parts of the Earth rotating around, underneath and through the tidal bulges. In particular, since there are two tidal bulges, a given point on the Earth's surface will revolve around through this relatively fixed pattern of high- and low- water, passing alternately through high- and low-tide regions.

Of course, how one describes this is going to be reference-frame dependent. From the point of view of the Earth itself, there are two big tidal bulges which race around and around and around, trying to keep up with the Moon (and the point in the sky opposite the Moon, respectively) as it rises and sets each day.

Let's see if we can now calculate the actual *amplitude* of the tides, i.e., the difference

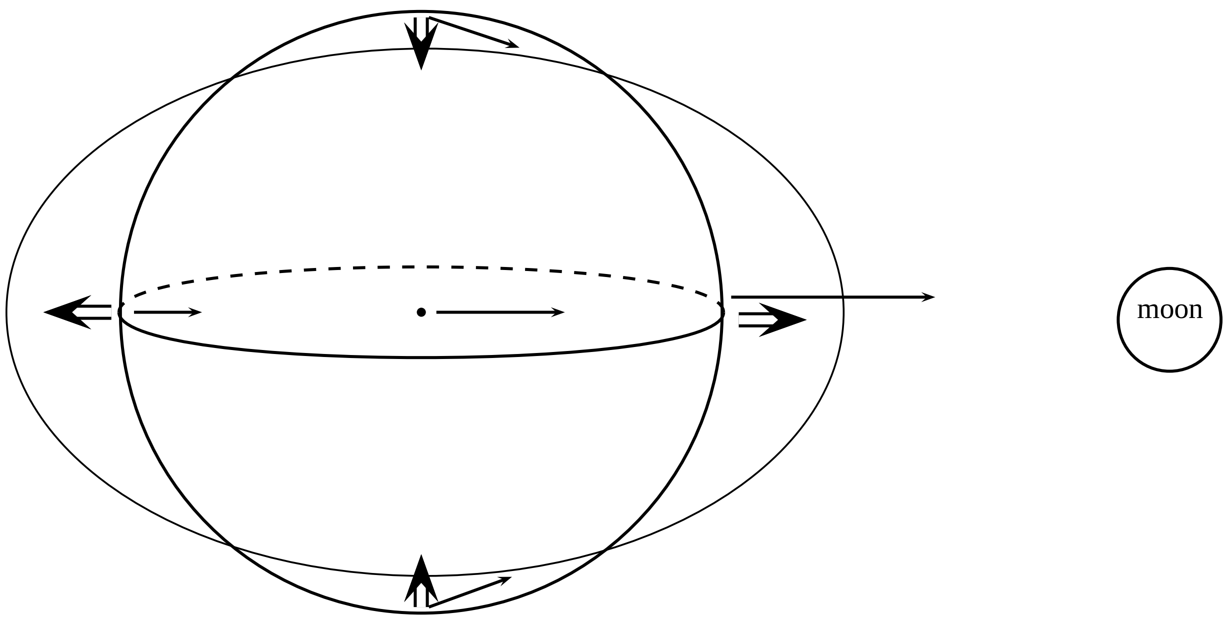

Figure 5.5: The tidal forces produced by the Moon on the Earth. The single arrows represent the gravitational force (per unit mass) exerted on a given part of the Earth. The double arrows represent the *difference* between the actual force acting at a point and the average force. This difference is called the "tidal force." The decrease of the Moon's gravitational influence with distance explains why the tidal force is toward the Moon on the right and away from the Moon on the left. The fact that the force is always directed straight toward the Moon explains the slight "tilt" of the forces at the top and bottom of the figure, which in turn results in a tidal force that points back in toward the center of the Earth. The net result of these differential forces is that water flows toward the two high-tide points on the right and left of the diagram, as indicated by the (much exaggerated) elliptical surface shown. Note finally that the three dimensional shape generated by these tidal forces will have rotational symmetry about the Earth-Moon axis. So it is low-tide not only at the top and bottom of the Earth (as shown in the figure) but also on the parts of the Earth that come out of the page, and the parts that go into the page.

h in height between the high- and low-tide points shown in the previous Figure. As with the calculations of the amplitude of the Equatorial bulge in the previous section, there are several ways to do this. The simplest is probably to use the equilibrium argument which says that the total energy change associated with moving (say) some mass m blob of water from the high-tide surface to the low-tide surface should be zero. You can work it out this way in the Projects. Here we'll adopt the slightly less straightforward, but in some ways more revealing, method of calculating first the effective gravitational accleration at different points on the Earth's surface, and then using this to calculate the "slant" of the equilibrium surface relative to "true horizontal" – just as we did in the previous section as a final way to analyze the oblateness of the Earth.

Figure 5.6 shows a cross-section of the Earth. We assume, to begin with, that the Earth is spherical. This seems like a dubious assumption given the previous section, but all we are going to calculate here is the *extra* deviation to the Earth's shape produced by the tidal interaction with the Moon. The idea is that the Earth's Equatorial oblateness is caused by its (daily) rotation – so we can ignore both the rotation and the oblateness in order to isolate the effect of the tides.

The Figure shows the two relevant contributions to the effective gravitational accleration, g_{eff}, at a point near the Earth's surface that is an angle θ down from the Earth-Moon axis. One of the contributions, g_{moon}, is of course the result of the Moon's gravitational influence. The other, g_c, is equal to the "average" (centripetal) acceleration of the Earth (toward the Moon, or, equivalently, about the Earth-Moon center of mass point).

The clearest way to think of this is to assume that we are using a non-inertial reference frame that is attached to the Earth. Since the Earth as a whole accelerates to the right in the Figure (if one uses an inertial frame), there will exist (in this Earth-attached frame) a "fictitious" gravitation-like force that pushes everything back to the left with a force proportional to its mass: $F_c = mg_c$. Note that the magnitude of g_c is just the accleration of the Earth (as a whole, on average) toward the Moon: $g_c = GM_{moon}/r^2$.

It is important here to appreciate that this is a non-inertial *but also non-rotating* reference frame. Using a rotating reference frame (e.g., centered at the Earth-Moon center of mass point and rotating in tandem with those bodies' mutual orbits) to analyze this problem is certainly possible. You can work it out in the Projects. But it can be a little confusing because, really by definition, the effect we are here trying to isolate and understand – the effect of the Moon's tidal forces on the shape of the Earth – has absolutely nothing to do with rotation. As we saw in the last section, rotation produces relatively *large* deviations from perfect sphericality, on the order of tens of kilmeters. The tides, of course, are nowhere near that high! (Luckily!) So we need to be careful to isolate the purely tidal effects we are interested in, by systematically avoiding any assumption (which may creep into the analysis if we're not careful to avoid it) that the Earth is rotating. So for now we forget about the rotational/spin motion of the Earth, and treat it as having a fixed orientation with respect to the fixed stars. Then, a reference frame that is rigidly attached to the Earth will be accelerating (because the Earth accelerates toward the Moon) *but not rotating.* And so the fictitious forces needed to use this non-inertial reference frame will be as described in the previous paragraph.

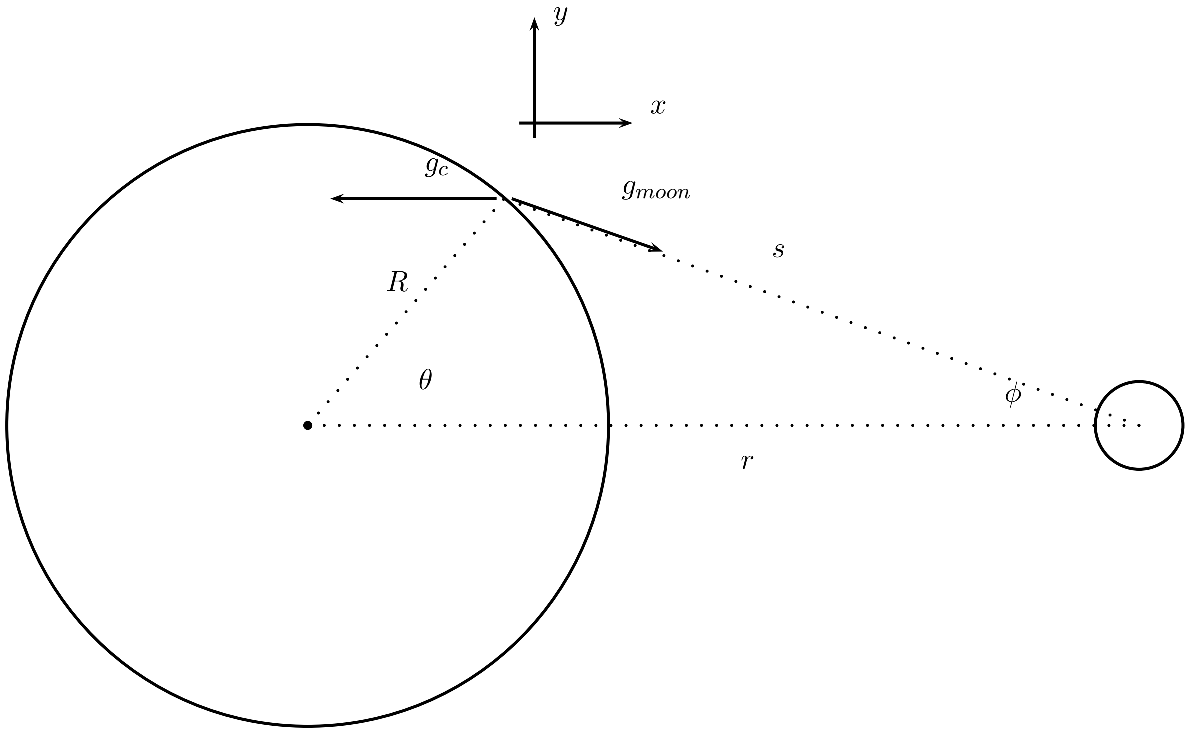

Figure 5.6: Diagram for calculation of tidal contributions to g_{eff} near the surface of the Earth.

The Figure also indicates an $x - y$ coordinate system, which will help us in writing down expressions for the x and y components of these two contributions to the effective gravitational acceleration for a point a distance R from the origin:

$$g^x_{eff} = \frac{GM_{moon}}{s^2}\cos(\phi) - g_c \tag{5.45}$$

$$= \frac{GM_{moon}}{r^2 + R^2 - 2Rr\cos(\theta)} - \frac{GM_{moon}}{r^2} \tag{5.46}$$

and

$$g^y_{eff} = -\frac{GM_{moon}}{s^2}\sin(\phi) \tag{5.47}$$

where we have used the law of cosines: $s^2 = r^2 + R^2 - 2Rr\cos(\theta)$.

Now we are going to make some simplifying approximations. The idea is essentially to expand things in the small parameter $R/r \approx 1/60$ and keep only the leading non-vanishing contributions in each term. As suggested (but understated) in the Figure, the angle ϕ is already small: $\sin(\phi) \approx R\sin(\theta)/r$. So we can then make the crudest possible approximation for the denominator in the expression for the y-component: $s^2 \approx r^2$. This gives

$$g^y_{eff} \approx -\frac{GM_{moon}R\sin(\theta)}{r^3}. \tag{5.48}$$

On the other hand, $\cos(\phi) \approx 1$. So we need to be more careful to pick off the similarly-sized contribution in the expression for the x-component. In particular, we'll ignore the R^2 – but not the $2Rr\cos(\theta)$ term – in the denominator. This gives

$$g^x_{eff} \approx \frac{GM_{moon}}{r^2 - 2Rr\cos(\theta)} - \frac{GM_{moon}}{r^2} \tag{5.49}$$

$$= \frac{GM_{moon}}{r^2}\left[\left(1 - \frac{2R}{r}\cos(\theta)\right)^{-1} - 1\right] \tag{5.50}$$

$$\approx \frac{2GM_{moon}R\cos(\theta)}{r^3} \tag{5.51}$$

which is indeed the same order of magnitude as the y component.

It will eventually be useful to have an expression for the Moon's tidal force not just on the surface of the Earth, but at an arbitrary location. The previous expressions can be easily converted by replacing $R\cos(\theta)$ with x, and $R\sin(\theta)$ with y:

$$g_e f f^x(x, y) \approx \frac{GM_{moon}}{r^3} 2x \tag{5.52}$$

and

$$g_e f f^y(x, y) \approx \frac{GM_{moon}}{r^3} y. \tag{5.53}$$

So those are the x and y components of the tidal force (per unit mass... probably we should say the tidal acceleration). As with the calculation of the size of the Equatorial bulge, however, it's really the *horizontal* component that directly affects the "slant" of the equilibrium ocean surface at angle θ relative to "true horizontal". It is easy enough to work out that

$$g^{horiz}_{eff} = g^x_{eff}\sin(\theta) - g^y_{eff}\cos(\theta) \tag{5.54}$$

$$= \frac{3GM_{moon}R}{r^3}\cos(\theta)\sin(\theta) \tag{5.55}$$

and hence that the angle made by the water surface at angle θ relative to "true horizontal" will be

$$\alpha = \frac{g^{horiz}_{eff}}{g} = \frac{3M_{moon}R^3}{M_{earth}r^3}\cos(\theta)\sin(\theta). \tag{5.56}$$

And, still just following the earlier calculation, this means that, over a small angle $d\theta$ at angle θ, the height of the water (relative to the initial spherical shape, i.e., constant height) will decrease by

$$dh = \alpha R d\theta = \frac{3M_{moon}R^4}{M_{earth}r^3}\cos(\theta)\sin(\theta)d\theta. \tag{5.57}$$

We need only finally integrate this from $\theta = 0$ to $\theta = \pi/2$ to find the total difference h between the heights of the low- and high-tide points:

$$h = \int dh = \frac{3}{2}\frac{M_{moon}}{M_{earth}}\frac{R^4}{r^3}. \tag{5.58}$$

If we plug in the actual values for the Moon's and Earth's masses and the relevant distances, we find

$$h = 54\,\text{cm} \tag{5.59}$$

or about two feet – certainly in the ballpark of the actual variations observed.

Actually, though, it is not at all uncommon for high- and low-tides to differ by two or three times this estimate, or more. The reason for this can be qualitatively understood by thinking again about what's happening using a reference frame that co-rotates with the Earth. Then the story one tells is that there are these two giant tidal waves which are constantly propagating around the Earth to the west. If the whole surface of the Earth were covered with water, the tidal bulges would more or less just flow around, and the above calculation would be pretty accurate. But, of course, they can't – there's *land* in the way! So, for example, the tidal bulge in the Atlantic Ocean runs up pretty hard against the whole east coast of the Americas, and has to somehow go *around* that land mass to get, just a few hours later, into the Pacific. So there is a tendency for the water to pile up more along the east coast than it would if there were no land there, much as a small ripple in the bathtub can make the water level go up and down with significantly greater amplitude when the ripple sloshes against the edge of the tub. And then, after all that water races around the continents into the Pacific, the two giant streams (from the north and south) meet in the middle and again create a bulge with an even-greater-than-equilibrium height. Of course, the details of this are *extremely* complicated and vary significantly between different geographical locations, even locations that are relatively close together. So if you want to know exactly when it will be high- or low-tide at a given location on a given day, consult a tide table! These are based on empirical fits to historical data, and so are much more reliable than any possible calculation a physicist could make. On the other hand, if you want to really understand what produces the tides and how to think about them, well, now you do!

There are several additional points that should be mentioned. First, although we've talked as if the tides are produced exclusively by the gravitational influence of the Moon, everything we've said applies equally much to the Sun. What low-high-tide difference h would be produced by the Sun (if we could isolate its tidal effect)? We can immediately co-opt our previous result, just changing everywhere the word "Moon" to "Sun" and re-interpreting the r to mean now the distance between the Earth and the Sun:

$$h_{Sun} = \frac{3}{2}\frac{M_{sun}}{M_{earth}}\frac{R^4}{r^3} = 25\,\text{cm} \tag{5.60}$$

which turns out, by sheer coincidence, to be of the same order of magnitude as the h produced by the Moon.

Of course, the Sun and Moon are both always present and always influencing the Earth's waters. The interesting point is that, depending on their relative alignment, the Sun and Moon can produce particularly *strong* tides, or particularly weak tides. Consider for example New Moon – when the Moon and the Sun are both in (roughly) the same direction relative to Earth. Then the tidal bulges produced by the two bodies are right on top of each other (one bulge on each side of the planet), and their amplitudes

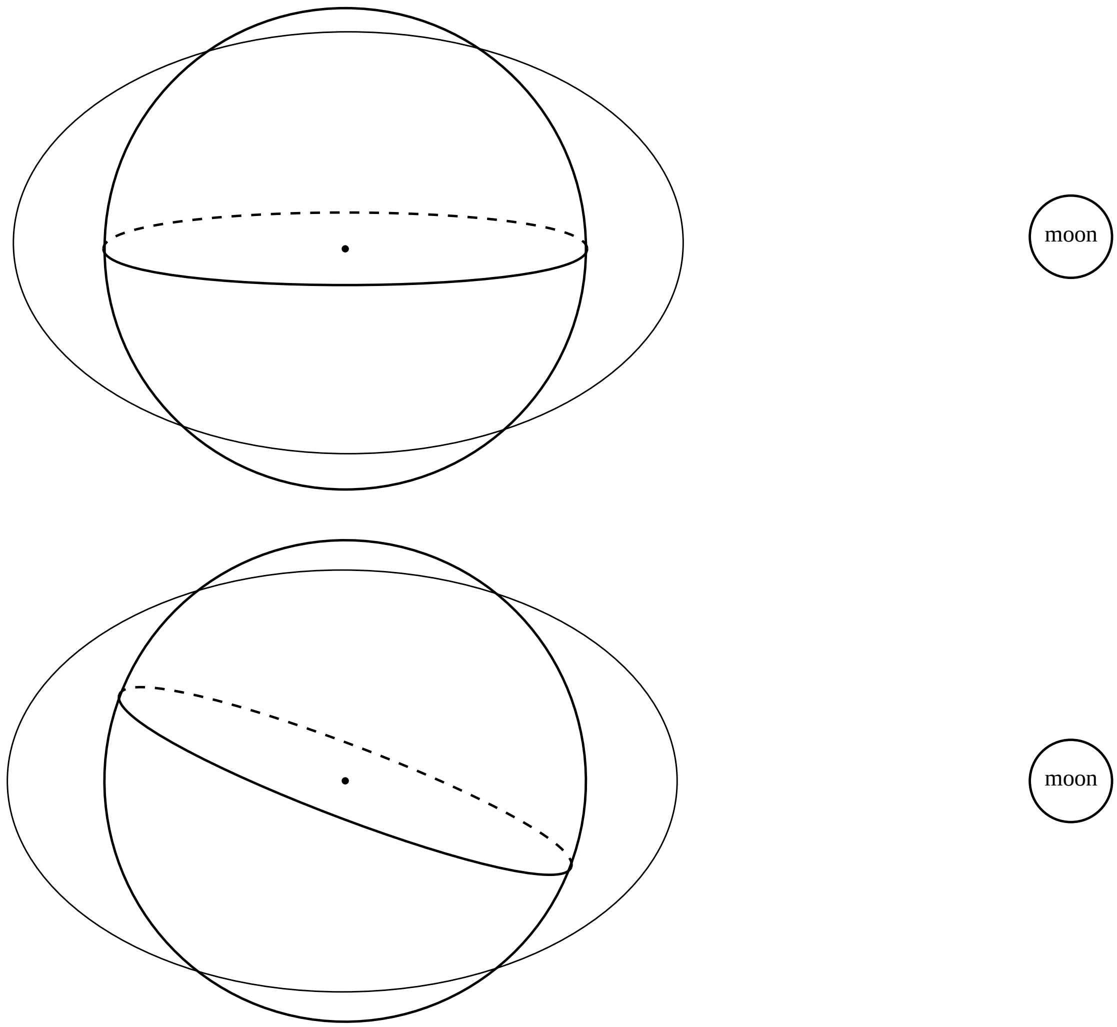

Figure 5.7: Two possible orientations of the Earth's tidal bulge relative to its geography. In the top panel, the Moon lies in the plane of the Earth's equator. An observer on the Equator will experience two equally high and two equally low tides on this day. Observers at other location will also experience two equally high and two equally low tides, but they won't be as high and low (respectively) as at the Equator. In the bottom panel, the Earth's spin axis is tilted down toward the Moon, so the two tidal bulges are somewhat north and south of the Equator, respectively. On this day, an observer at the Equator will observe two equally high high tides and two equally low low tides (but they won't be as high and low, respectively, as they were on the day pictured in the top panel). An observer at moderate latitude will observe two high tides, but one of them will be considerably higher than the other. The two low tides will be about the same. Observers at very extreme latitudes (near the north or south Poles) may experience just one high tide and just one low tide on this day! That should give you a sense of the monthly variations that are possible in the tides, and how those variations vary by latitude. The *seasonal* variations mentioned in the text arise the same way, but with the Sun replacing the Moon in the Figure.

add. Hence, one expects particularly strong tides (meaning particularly high high-tides and particularly low low-tides) around New Moon – and, as you can see with a little thought, also around Full Moon. By contrast, when the Moon is about half full, so the Sun, Moon, and Earth make a right triangle, the Moon tries to create a high tide at the same place (on Earth) that the Sun tries to create a low tide, and vice versa. That is, their effects tend to *cancel* resulting in particularly weak tides (not too high high-tides and not too low low-tides). The weak tides that occur when the moon is half full (either waxing or waning) are called "neap tides". The strong tides that occur at Full and New Moon are called "spring tides" – not because they happen in the spring, but, evidently, because the waters spring up particularly high then.

There are, however, some seasonal (and monthly) variations in the tides as well, which have to do with the fact that the Earth's spin axis is not perpindicular to, but tilted relative to, the plane of the ecliptic. See Figure 5.7. Additional seasonal and monthly variations are produced by the fact that neither the Moon's orbit around the Earth nor the Earth's around the Sun are circular. Instead, as discovered first by Kepler, the orbits are slightly eccentric ellipses. The distance r between the Earth and the Moon, for example, varies up and then down away from its average by about 5 % each month. And since the strength of the tides depends on this distance to the *third power*, the relatively small changes in the distance to the Moon can produce relatively large changes (15-20% variations away from average) in the strength of the tides. (The same is also true for the Earth's orbit around the Sun, but since this is eccentric only by one or two percent, the corresponding seasonal variations in the Sun's tidal influence are smaller.)

So next time you visit the ocean, pay attention to the tides. In particular, notice how the rising and falling of the tide correlates with the location and phase of the moon.

5.3 The Non-Spherical Earth and Associated Torques

As we mentioned in passing in Chapter 4, a *uniform* gravitational field – like that near the surface of the Earth – will exert a net force *but no net torque* on an object, no matter how complicated its shape. On the other hand, a *non-uniform* gravitational field – like the spherically-symmetric radially-inward field produced by a moon or planet or star – can exert not only a net force but also a net torque on an arbitrarily shaped object. Consider, for example, the situation depicted in Figure 5.8.

It can be shown (we won't bother here) that a spherically symmetric object, however, cannot have such a gravitational torque exerted on it. (Actually, it's sort of the converse of the earlier proof that a spherically symmetric body acts, gravitationally, just like a point mass – the point here is that such a body also *re*-acts, gravitationally, just like a point mass.) In order for a gravitational torque to be produced on an object, the object must lie in a non-uniform gravitational field and must itself be non-spherically-symmetric. Of course, just like the imaginary giant barbell in the Figure, the Earth sits in the not-quite-uniform gravitational field of the Moon. (The tidal forces we analyzed in the previous section can be thought of as nothing but the departures of the Moon's gravitational field from uniformity in the vicinity of the Earth.) Moreover, both of the last two sections have concerned themselves with respects in which the Earth fails to

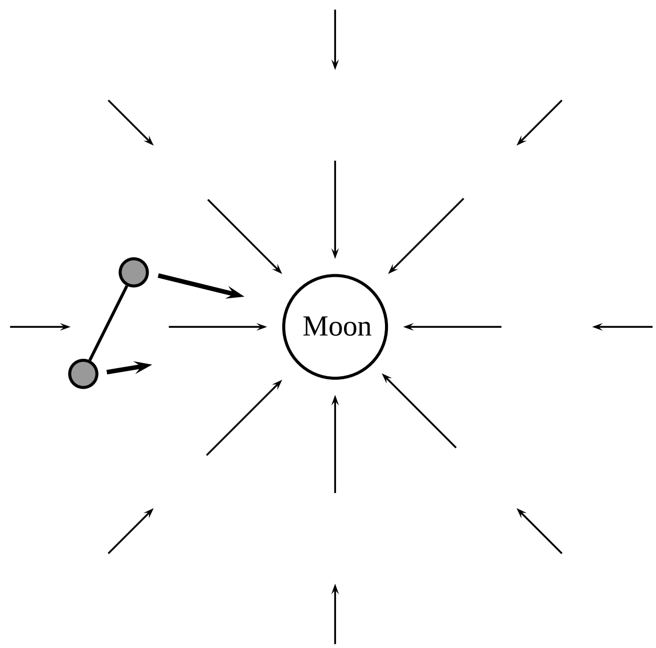

Figure 5.8: A giant barbell in space near (say) the Moon. The net gravitational force on the barbell (the sum of the two forces exerted on the two masses) pulls it toward the Moon. But because the gravitational field produced by the Moon is not uniform, the two masses composing the barbell have different forces exerted on them – which produces, in addition to the overall tendency to accelerate toward the moon, a *torque* which tends to rotate the barbell (clockwise in the Figure).

be perfectly symmetric. So one should expect that the Earth's various bulges result in *torques*, which – in some ways we'll now explore – affect the rotational state of the Earth.

5.3.1 The Tidal Torque

In the previous section, we discussed how the non-uniformity of the Moon's gravitational field near the Earth (the tidal forces) produces two tidal bulges, one on the side facing the Moon and the other on the opposite side. We calculated the equilibrium height of the bulges and discussed the simple equilibrium model in which a given point on the Earth's surface just rotates around, moving alternately through the high- and low-tide regions, and thus experiences two high- and two low-tides per day.

There is also, however, an important dynamical coupling between the rotation of the Earth and the tidal bulges. As viewed from (say) an inertial reference frame above the Earth-Moon system, the tidal bulges have to move (pretty fast!) relative to the rotating Earth, just to stay in their equilibrium positions. And because there is some friction between the solid rotating Earth under the oceans, and the waters themselves, the tidal bulges don't *quite* keep up. Put another way, the rotation of the Earth is

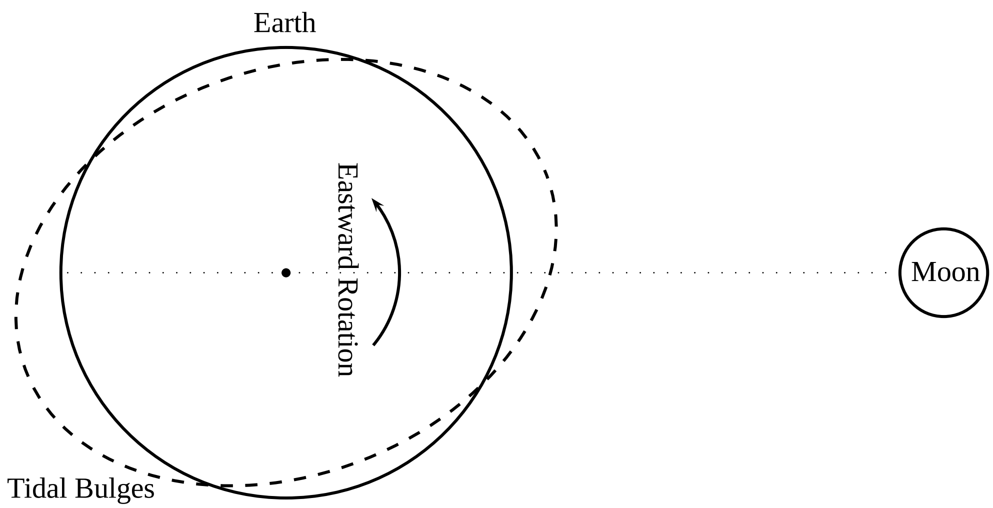

Figure 5.9: The perspective here is looking down from far above the North Pole. The Earth's daily Eastward Rotation drags the tidal bulges a little bit to the east relative to their equilibrium positions (which would be along the line connecting the Earth's and Moon's centers). (Note that the picture exaggerates this. In fact, the tidal bulges are only a few degrees to the east of the Earth-Moon line.) The tidally bulging Earth therefore acts just like the giant barbell from the previous Figure: the bulge that is closer to the Moon is attracted toward the Moon more strongly than, and in a slightly different direction from, the bulge on the other side, producing a torque that tends to turn the Earth to the west. Or here is a slightly better way to think about the same thing. If one imagines the slightly-rotated tidally bugling Earth superimposed on the tidal forces as shown in Figure 5.5, the result is clearly a net torque: both bulges are pulled, by those tidal forces, in a way that tends to produce a clockwise (i.e., westerly) rotation of the bulging Earth. Of course, since the Earth is already rotating to the East, the result of this torque is a (very gradual) decrease of the easterly angular velocity. That is, because of the torque exerted by the Moon on the tidally bulging Earth, the length of the day is very gradually increasing!

constantly pulling the bulges away from their equilibrium positions (just under the Moon and opposite it). The result is that the tidal bulges are not precisely in their equilibrium positions, but are instead pulled a few degrees to the east by the rotating Earth. See Figure 5.9.

As explained in the Figure's caption, the Moon's tidal forces produce a net torque on the Earth because of this slight departure of the tidal bulges from their equilibrium orientation. This torque acts to slow down the rate at which the Earth rotates, i.e., to increase the length of the day. Indeed, scientists have measured that the length of the day is increasing by about 1.6 milli-seconds per century.

It is very interesting to consider this process from the point of view of the Earth-

Moon system. For that system, all of the complicated frictional and tidal/gravitational forces that are involved in the slowing down of the Earth's rotation are *internal forces*, which can therefore produce no *net* torque. The total angular momentum of the Earth-Moon system must therefore be a constant – which means that, since the (eastward) spin angular momentum of the Earth is *decreasing*, the (eastward) orbital angular momentum of the Moon must be *increasing*. It can be shown that the orbital angular momentum for a roughly circular orbit is proportional to the square root of the radius of the orbit – so increasing orbital angular momentum implies increasing radius. Thus, it follows from the conservation of angular momentum that the size of the Moon's orbit should be slowly increasing.

Amazingly, this too has been directly measured in recent decades. When the Apollo astronauts landed on the Moon in the early 1970s, they left some mirrors (technically "corner reflectors") from which Earth-based scientists can reflect light. Measuring the amount of time it takes for a pulse of light shot toward the Moon to be reflected and subsequently detected (and knowing the speed of light) allows for extremely precise measurements of the distance to the Moon. And indeed, this distance has been *measured* to be increasing at a rate of approximately 3.5 cm per year.

By the way, the reason for this gradual (but measurable) change in the Moon's orbit can be understood without mentioning angular momentum conservation. As we have seen, the torque on the tidally bulging Earth can be understood as a result of the bulge closest to the Moon being attracted to the Moon more strongly than the bulge on the far side. But also vice versa: the tidal bulge closest to the Moon *attracts* the Moon more strongly than does the bulge on the far side. So the net force exerted by the Earth on the Moon is not quite toward the center of the Earth, but rather ever-so-slightly tilted toward the Moon's direction of motion. This component does positive work on the Moon, increasing its energy and allowing it to "climb" into ever-higher orbits.

The upshot of all this is that the Earth-Moon system is not in equilibrium. The Earth's "daily" rotation rate is decreasing, and the Moon's orbital radius is increasing. When and how will these gradual changes cease? A little thought reveals the answer: when the Earth's (spin) angular velocity matches the Moon's (orbital) angular velocity. In other words: when the Earth daily rotation slows so much that it always presents the same face to the Moon. Then the Moon (which *already* always presents the same face to the Earth) and the Earth will be "tidally locked" in a face-to-face dance.

So – as a result of the subtle interplay of tidal forces, friction between the oceans and the sea floor, and the laws of rotational dynamics – your distant ancentors may someday be able to see the Moon in the sky all the time (or never, depending on where they live).

5.3.2 Torque on the Equatorial Bulge

We previously treated the Earth's Equatorial Bulge as an intrinsically interesting feature that can be understood and explained using Newton's theory of gravitation and some concepts of rotational dynamics and kinematics. But the Earth's Equatorial bulge is interesting for another reason, too: just as with the tidal bulges, the tidal forces exerted by the Moon (and Sun) interact with the Equatorial bulge to produce a torque. And this torque, like the one on the tidal bulges, results in some interesting gradual changes in the Earth's state of rotation.

Actually, the effect of the torque (exerted jointly by the Sun and Moon) on the Equatorial bulge is something we've already discussed, way back in Chapter 1: the precession of the equinoxes – that subtle long-period turning of the Earth's rotation axis that was (despite its roughly 26,000 year period) noticed already by the Ancient Greeks.

In principle, the mechanism here is simple. The Earth spins, like a top. And – *because the Earth is not quite perfectly spherical* – the Sun and Moon exert a gravitational torque on the Earth. The torque is produced by the Sun's and Moon's tidal forces, which would tend to align the Earth's Equatorial plane with the plane of the Ecliptic (the plane of the Earth's orbit around the Sun, roughly also the Moon's orbit around the Earth). But just as the gravitational torque on the spinning top causes it to precess rather than tip over, so with the Earth: the gravitational torques exerted on its bulging Equator by the Sun and Moon cause its spin (or spin angular momentum) axis to sweep out a cone, always staying roughly the same 23.5° away from the fixed point among the stars called the Pole of the Ecliptic.

Really, the story here is precisely like the story from the previous chapter for the top. So there are only two things to fill in. First: why and how, exactly, does the Moon or Sun exert a net torque on the Earth? And second: how big is that torque, and does it – in accordance with Equation 4.102 – account for the observed rate of one revolution per 26,000 years?

We'll discuss the first point here and then leave the second part (a hard but very cool calculation) for the Projects.

Actually, there's not that much to say since the effect is the same as that for the tidal bulges. See Figure 5.10 for a sketch of the Equatorially bulging Earth sitting in the tidal force field produced by (say) the Moon.

The only subtlety is that, since the orientation of the Earth's spin axis is (approximately!) the same throughout the month or year, the tidal forces and bulge will not always be exactly as depicted in the Figure. It is probably easiest here to think first about the tidal forces exerted by the Sun. Then, Figure 5.10 will depict the situation correctly at the Summer Solstice (with the Sun to the left along the negative x-axis) and also the Winter Solstice (with the Sun to the right along the positve x-axis). These two times turn out to correspond to the torque being a *maximum*. And it is important that at these two times the torque is in the same *sense*, the same direction.

Around the equinoxes, however, the situation is rather different. The relevant tidal forces are as shown in Figure 5.11. As should be clear qualitatively from the Figure, the torque now *vanishes*. Hence, over the course of the year, the torque exerted by the Sun on the Equatorially bulging Earth varies back and forth (twice) between some maximum value and zero.

Since this back and forth variation in the torque turns out to be extremely fast compared to the main effect produced by the torque (the 26,000 year period precession of the equinoxes), it is reasonable to calculate an average torque, and then treat the phenomenon as if that average torque were exerted steadily in time. We may guess that the average torque produced by the Sun's tidal forces will be about half of the maximum

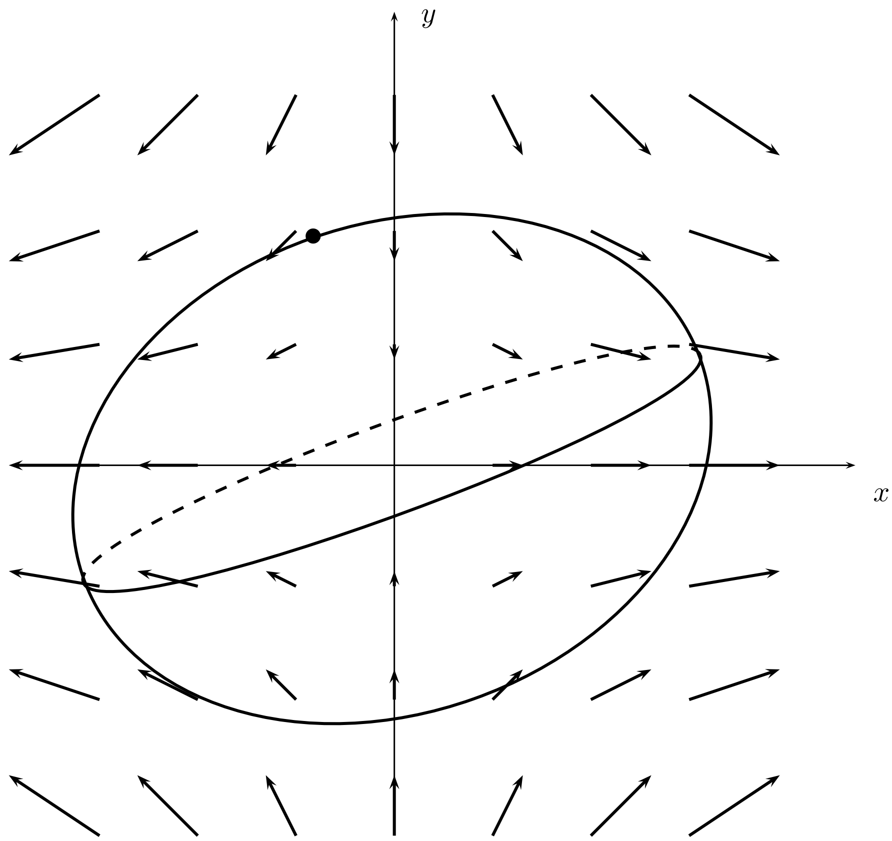

Figure 5.10: The Equatorially bulging Earth lies in the tidal force field produced by (say) the Moon. The picture will be accurate if the Moon is to the right along the x-axis, or to the left along the negative x-axis. Clearly the tidal force on the bulge on the right tends to turn it clockwise, as does the tidal force on the bulge on the left. There is therefore a net torque exerted on the Earth by these tidal forces.

torque (exerted at the Solstices):

$$\tau_{Sun}^{avg} \approx \frac{1}{2} \tau_{Sun}^{max}. \tag{5.61}$$

Now finally note that everything we've just said about the tidal forces and torques produced by the Sun, applies in just the same way to the Moon. The only difference really is that we don't have terms for the points in the Moon's orbit around the Earth which correspond to the Solstices and Equinoxes – i.e., the times when the Earth's spin axis is tilted maximally toward or away from the Moon (the "Lunar Solstices") or tilted down perpindicularly from the Moon-Earth line (the "Lunar Equinoxes"). So it would have been a little harder to describe and understand. But if you followed the discussion for the Sun, everything is truly the same for the Moon – except that the relevant torque (produced by the Moon's tidal forces on the Earth's Equatorial bulge) oscillates back

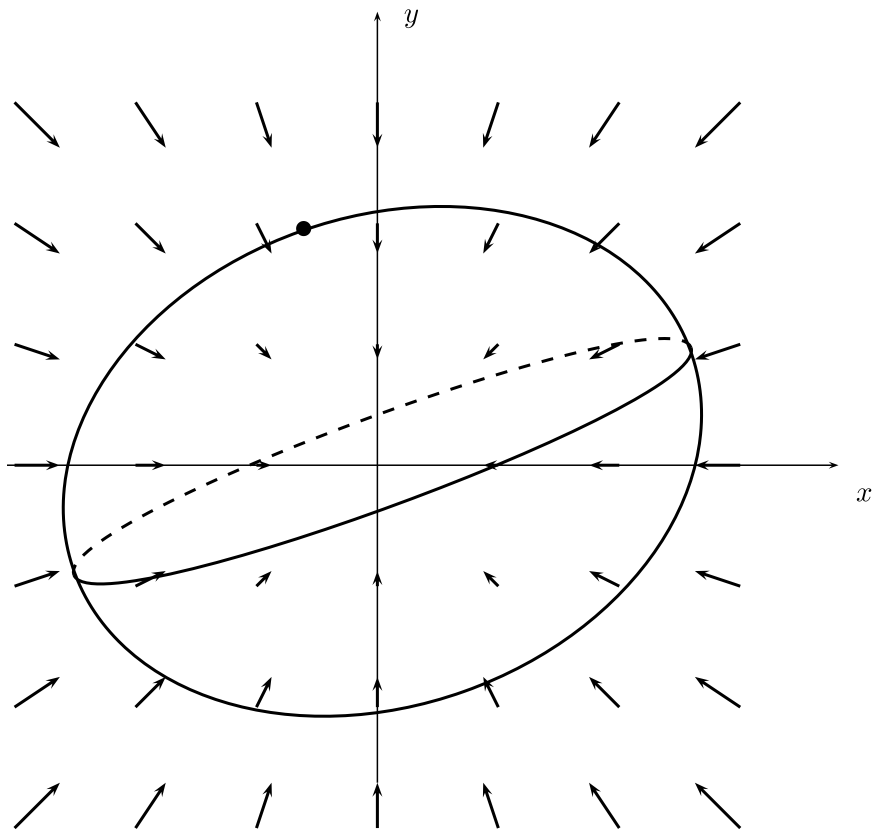

Figure 5.11: The Earth sitting in the tidal force field produced by the Sun around the Spring or Autumn Equinox. The perspective is (say) from the Sun. The tidal forces are radially symmetric in a plane perpindicular to the Earth-Sun line, and so produce no torque on the Earth – or at least, it is clear from the picture that there is no torque on the particular planar slice of Earth shown. But all of the other slices will have the same symmetry pattern, and so, indeed, the total torque will vanish.

and forth between its maximum value and zero twice per month, rather than twice per year. But we still have, in analogy with the previous equation, that

$$\tau_{Moon}^{avg} \approx \frac{1}{2}\tau_{Moon}^{max}. \tag{5.62}$$

The time-averaged *total* torque exerted on the Earth (in virtue of its Equatorial bulge) is therefore

$$\tau_{total}^{avg} = \tau_{Sun}^{avg} + \tau_{Moon}^{avg} = \frac{1}{2}\left(\tau_{Sun}^{max} + \tau_{Moon}^{max}\right). \tag{5.63}$$

So if we can calculate (or approximate) the "max" torque produced on the Equatorial bulge by the Sun at the Solstices – and by the Moon at the "Lunar Solstices" – we'll be able to plug the resulting total torque into Equation 4.102 and see if, indeed, this process accounts quantitatively for the observed precession rate. But we'll leave that fun project for the Projects.

5.4 Measuring Masses

Back in Chapter 4, we discussed the Cavendish experiment in which Newton's gravitational constant G was first measured. Because the gravitational acceleration $g = 9.8\,m/s^2$ of objects near the Earth's surface is readily measureable, and because this acceleration is given, according to Newton's theory, by

$$g = \frac{GM_{earth}}{R_{earth}^2} \tag{5.64}$$

– and because the radius of the Earth is also known – the measurement of Newton's constant G allows the mass of the Earth to be computed. This is why, as we discussed, this laboratory measurement was and is often referred to as a means of "weighing the Earth."

We also discussed, in that earlier chapter, how a similar approach could be used to determine the mass of the Sun. Since, for example, the (centripetal) acceleration of the Earth toward the Sun is known

$$a_{earth} = \omega^2 R = \frac{2\pi^2 R}{T^2} = \frac{4\pi^2 \times 1\,AU}{(1\,year)^2} = .0059\,m/s^2 \tag{5.65}$$

and given, according to Newton's theory, by

$$a_{earth} = \frac{GM_{sun}}{R^2} \tag{5.66}$$

where $R = 1\,AU$ is the Earth's orbital radius, the mass of the Sun can be worked out:

$$M_{sun} = \frac{R^2 a_{earth}}{G} = \frac{4\pi^2 R^3}{T^2 G} = 2 \times 10^{30}\,kg \tag{5.67}$$

or about 300,000 times the mass of the Earth.

Here is the principle involved: whenever a relatively light body moves under the gravitational influence of a relatively heavy body, and the relevant kinematical properties of the light body (its acceleration toward and distance from the heavy body) can be measured directly, the mass of the heavy body can be inferred. This is a relatively simple point, but an extremely important and fruitful one for modern astronomy and astrophysics. For example, it is by this same method that the masses of other planets can be determined – but only if those planets have *moons*!

Moons orbiting Mars, Jupiter, and Saturn were discovered when (or shortly after) Galileo first pointed a telescope to the heavens. Thus Newton, in the Principia, was already able to estimate the masses of these planets. A more recent and particularly interesting instance is the planet (recently demoted to "dwarf planet" status) Pluto, which was discovered in 1930. Pluto's mass, however, remained unknown until 1978, when a moon ("Charon") orbiting Pluto was discovered.

The apparent (angular) diameter of Charon's orbit, combined with knowledge of the distance to the planet-moon system, allowed the absolute size (R) of its orbit to be determined. Observations over time also allow the period of Charon's orbit to be determined. By plugging this information into Equation 5.67, Pluto's mass can be calculated. You can work through that calculation in the Projects.

We should note, though, that this is a bit over-simplified. The mass of Pluto turns out to be pretty low – so low that its moon, Charon, really is not "relatively light" compared to it. Indeed, it turns out that the center of mass of the Pluto-Charon system is not within Pluto's body at all, but is rather in the empty space between them. (By comparison, the center of mass of the whole solar system is somewhere inside the Sun – slightly toward Jupiter from its center, typically; similarly, the center of mass of the Earth-Moon system is within the Earth, some 1700 km below the Earth's surface, on the side facing the Moon, obviously.) The Pluto-Charon system is therefore sometimes classified as a "dwarf double planet" system rather than a (dwarf) planet plus a moon.

Another interesting (and only recently-discovered) fact about the Pluto-Charon system is that both bodies are "tidally locked" to one another. This is also a result of the fact that the two bodies are of comparable mass.

Anyway, the fact that the two bodies are of comparable mass – and hence must really be described as each orbiting around their mutual center of mass – requires a somewhat more careful analysis to convert the observed kinematical information into a determination of their masses. Let's work this out in general for two objects of mass m_1 and m_2, orbiting around their mutual center of mass with (circular) orbits of radii R_1 and R_2, as shown in Figure 5.12.

For the Pluto-Charon system, we would observe the system "edge-on" rather than the "face-on" perspective shown in the Figure. The latter, however, is a little simpler for analyzing the physics. In any case, no matter what perspective we have on the system, as long as we can observe it over time (and as long as the absolute distance to the system is known, so the apparent angular separations can be converted into absolute distances) we can determine the radii of the two orbits, R_1 and R_2. These are distances measured from the (empty) center of mass point, so one might wonder how this point can be located. The answer is simple: it is the center of the two observable orbits.

It follows from the definition of the center of mass that the product $m_1 R_1$ should equal $m_2 R_2$. This can be converted into an expression for the mass ratio:

$$\frac{m_1}{m_2} = \frac{R_2}{R_1}. \tag{5.68}$$

An additional algebraic constraint on the two masses can then be inferred from orbital dynamics. According to Newton's theory, the mass m_2 exerts on m_1 a force of magnitude $F = Gm_1 m_2/(R_1 + R_2)^2$ which produces acceleration $a_1 = Gm_2/(R_1 + R_2)^2$. But this is just the observed centripetal acceleration of m_1, so we may write

$$\frac{Gm_2}{(R_1 + R_2)^2} = \frac{v_1^2}{R_1} = \frac{4\pi^2 R_1}{T^2} \tag{5.69}$$

where T is the period of the orbit. The same reasoning leads to a corresponding condition

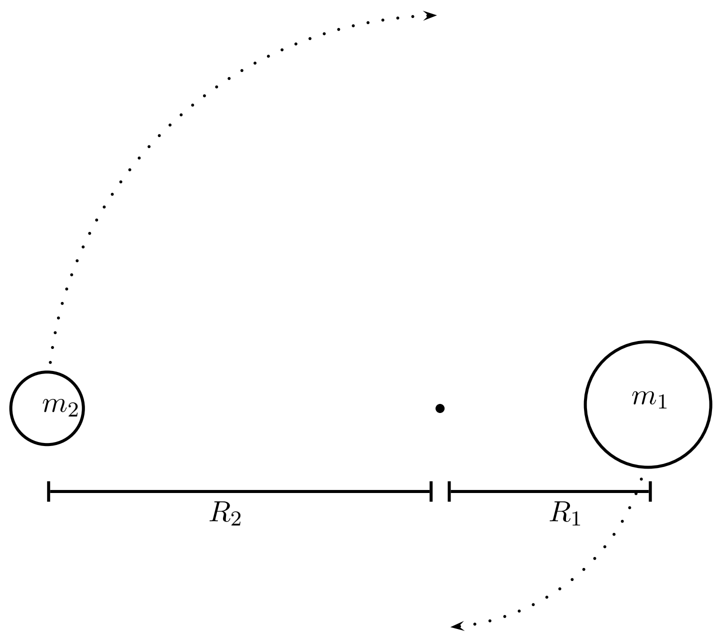

Figure 5.12: A binary system: two bodies, of mass m_1 and m_2 respectively, orbit about their mutual center of mass (the black dot in the Figure) with circular orbits of radii R_1 and R_2.

for the other mass:

$$\frac{Gm_1}{(R_1+R_2)^2} = \frac{4\pi^2 R_2}{T^2}. \tag{5.70}$$

The previous two equations can then be added, and the result simplified, to give an expression for the sum of the masses:

$$m_1 + m_2 = \frac{4\pi^2 (R_1+R_2)^3}{GT^2}. \tag{5.71}$$

With the ratio and the sum both determined by observable quantities, it is then clear that the two masses – m_1 and m_2 – can each be uniquely determined.

This would all probably be analytical overkill if it were useful only as a way to figure out the mass of Pluto and its moon Charon. But in fact this same technique can be used to determine the masses of *stars*, many of which, perhaps surprisingly, are found to be trapped in a gravitational orbit with another star – a so-called binary star system. For example, two of the best-known stars – Polaris (the north star) and Sirius (the bright star near Orion) – happen actually to be members of binary star systems.

The simplest kind of case is a binary star system in which the two stars are individually observable, such that their individual orbits can (as in the case of Pluto and Charon) be tracked over time. If the absolute distance to the binary star system can also be determined, it is then straightforward to measure R_1, R_2, and T from observation, and hence to infer (just as sketched above) the masses of the two stars.

Actually, it is more commonly successful to measure the masses of stars in binary systems in a slightly more subtle way. This takes advantage of the so-called Doppler effect, which is probably familiar in the case of sound: the paradigm example is the ambulance siren that sounds higher in pitch as the ambulance approaches you, but then appears to drop in pitch as the ambulance passes you and starts to recede. The physics involved here is that the *observed* frequency f of a wave (such as the sound wave emitted by the ambulance siren) depends not only on the *intrinsic* frequency of the source, f_0, but also on the radial velocity, v_r, of the source – i.e., the rate at which its distance from the observer is decreasing. For a sound wave, the relevant formula for the Doppler shift is

$$\Delta f = f - f_0 = f_0 \frac{v_r}{c} \tag{5.72}$$

where c is the speed of sound. For a *light* wave, the formula is the same (at least as long as v_r is small compared to c), but with c now the speed of light: $c = 3 \times 10^8\, m/s$.

The upshot is that, by carefully monitoring the *frequency* of light emitted by stars, one can learn something about the speed with which they move, toward or away from the observer. For stars in a binary system as discussed above – but viewed "edge on" – the radial velocity will oscillate back and forth (say, around zero) with a maximum absolute value

$$v^{max} = \omega R = \frac{2\pi R}{T} \tag{5.73}$$

where T is the period of the orbit and R is its radius.

The point is then that, for a so-called "spectroscopic binary" in which this Doppler wobble can be detected for both stars, we can rewrite the above mass-determination equations purely in terms of the radial velocity amplitudes, $v_1^{max} = \omega R_1$ and $v_2^{max} = \omega R_2$, instead of the radii R_1 and R_2 which are, as a matter of observational fact, much harder to measure than the velocities.

The only problem is that, if we just determine the v^{max} values from spectroscopic data without actually resolving the precise motion of the two stars, there is no way to know whether the binary system is being viewed precisely "edge-on." To be general, we should assume that the system is inclined at some angle i, in which case the maximum observed radial velocities are given by

$$v^{max} = \omega R \sin(i) = \frac{2\pi R}{T} \sin(i) \tag{5.74}$$

where $i = 0$ corresponds to the "face-on" perspective shown in the Figure and $i = \pi/2$ corresponds to the "edge-on" perspective. With this more general relationship, the relevant formulas for the masses of the two stars in the binary become

$$\frac{m_1}{m_2} = \frac{v_2^{max}}{v_1^{max}} \tag{5.75}$$

and

$$m_1 + m_2 = \frac{T}{2\pi G} \frac{(v_1^{max} + v_2^{max})^3}{\sin^3(i)}. \tag{5.76}$$

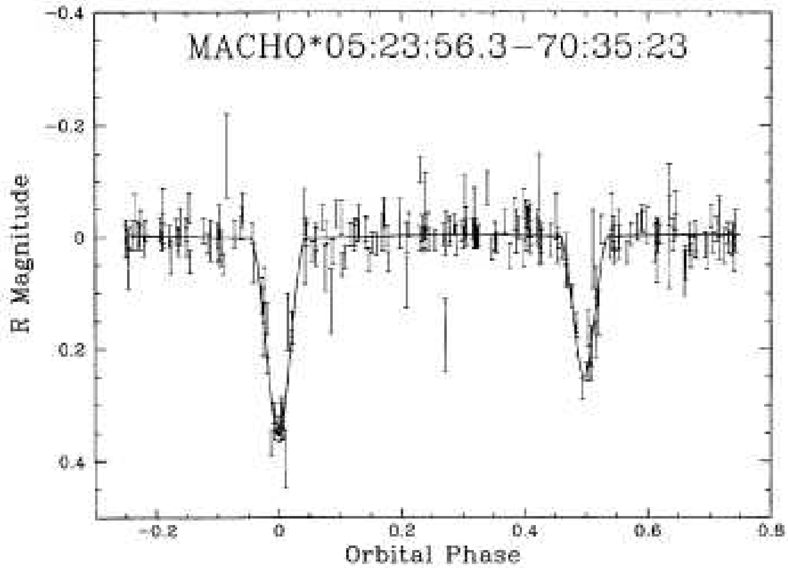

Figure 5.13: The "light curve" for an eclipsing binary star system. (A light curve is a plot of the intensity of light coming from a certain source, plotted against time. Since the light curve for this system is *periodic*, the intensity has been plotted against the phase of the period. This makes the structure of the periodic curve much clearer to the eye.)

In some cases, the two stars in a binary system can be observed to *eclipse* one another as they orbit. The light curve for one such eclipsing binary is shown in Figure 5.13. The eclipsing implies that the system is being observed edge-on, such that $\sin(i) \approx 1$. In such cases, careful observations of T, v_1^{max}, and v_2^{max} allow very accurate determinations of the masses of the two stars. In other cases, there is no way of determining i from the observations, and the most one can do is put a lower limit on the masses.

At this point a fair question would be: who cares about the masses of stars? Part of the answer would surely be that, as we now know, gravity plays a crucial role in the formation and evolution of stars. So if you want to understand stars – which means, if you want to understand the universe and our place in it – you better know something about the source of gravitation, which is mass. As just one concretization of this (perhaps otherwise unsatisfying) answer, note that empirical studies of relatively nearby binary star systems reveal an amazing correlation between stars' mass and luminosity. See Figure 5.14. "Luminosity" refers to a star's intrinsic brightness – the total amount of energy radiated, as light, per unit time. This can be determined by measuring the *intensity* of the star's light – that is, the energy per unit time passing through a unit area (e.g., a detector) here on Earth – and then multiplying by the area of a sphere

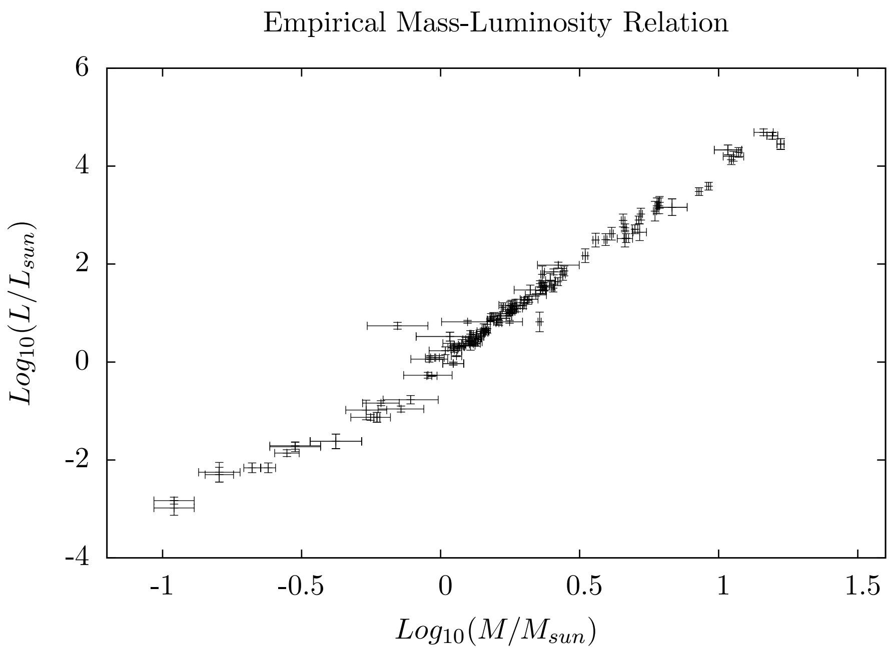

Figure 5.14: Graph of the mass-luminosity relation. Data are from Popper 1980 and are for "main sequence" stars only. See C/O problem 7.9 on page 200.

whose radius is the distance from that star to the Earth, the idea being that the star radiates its light uniformly in all directions, so the total amount of light (the luminosity) should equal the amount of light per unit area (as sampled over some very small area here on Earth) times the total area through which starlight of that intensity passes. As an equation,

$$L = I \times 4\pi D^2 \tag{5.77}$$

where L is the star's intrinsic luminosity, I is the measured intensity of its light here on Earth, and D is the distance to the star.

Anyway, the mass-luminosity correlation indicates that the directly observable features of stars (such as their brightness, but including also such features as color and radius) are intimately related to their intrinsic internal structures. Better understanding the details of this connection between the hidden internal structure of stars and their outward appearances is a major part of astrophysics. As you can imagine, this relies not only on the theory of gravitation, but also thermodynamics, hydrodynamics, optics, and even nuclear physics – because it is the nuclear process of *fusion*, occuring in the cores of stars, which fuels them.

One particularly interesting implication of the empirical mass-luminosity relation is that massive stars live much shorter lives. All other things being equal, one might have thought that a more massive star would burn longer than a less massive star, since it has more internal fuel. (The fusion reaction that powers stars is the nuclear "burning" of Hydrogen into Helium. More massive stars, however, will continue to burn after the Hydrogen is used up – by fusing Helium into Carbon, for example.) But the empirical relationship between mass and luminosity shows that all other things are not equal. A star that is, say, twice the mass of the Sun will be about 10 times brighter, and will therefore burn through its fuel in roughly a fifth the time. (The Sun will run out of Hydrogen fuel – and puff up into a red giant, before eventually settling back down to become a "white dwarf" – in about 5 Billion years.)

A star that is a hundred times the mass of the Sun will use up its fuel millions of times faster, and hence have a lifetime that is thousands of times shorter than the Sun. Such massive stars not only end their lives sooner than the Sun – they also end it much more dramatically. We will discuss in the next section.

5.5 Cataclysms

In our discussion of the Earth's tides above, we noted that, because the overall tidal effect depends on the third power of the distance between the central and orbiting bodies, relatively small variations in this distance (as result from the Moon's not-quite-circular orbit) can produce relatively large fluctuations in the strength of tidal effects. This is of course a general fact about tidal forces, which applies not just to the Moon's tidal influence on the Earth, but also the Earth's on the Moon, Pluto's on Charon, and so forth. Let us think about the following thought experiment in terms of a generic planet-moon system.

Imagine that some planet's moon was somehow brought into progressively smaller and smaller circular orbits around the planet. The planet's tidal effect on its moon would grow and grow, in accordance with the inverse-cube law just mentioned, and so – at least to the extent that the moon is deformable over the relevant timescales – its departure from sphericality would increase. But at some point (i.e., at some particular distance from the planet) a dramatic transition will have to occur: the tidal forces acting on the planet would become comparable in size to the (largely gravitational) forces by which the moon holds itself together as an integrated body. At this point, the moon would be unable to hold itself together, and would be literally torn apart by the tidal forces.

To estimate when this should happen, we may calculate the distance at which, say, a rock on the side of the moon facing the planet is pulled just as hard toward the planet (by the tidal force) as it is pulled toward the moon (by the moon's own gravitational force). The strength of the gravitational field produced by the moon's own gravity is

$$g_{Moon} = \frac{GM_{moon}}{R_{moon}^2} \tag{5.78}$$

while the tidal field exerted by the planet is (for a rock on the near side)

$$g_{tidal} = \frac{2GM_{planet}R_{moon}}{r^3} \tag{5.79}$$

where R is the radius of the moon and r is the critical planet-moon separation.

If the moon were just at this critical distance where the two previous expressions are equal, a rock released from just above the Moon's surface would be at a kind of unstable equilibrium point, and might either fall back to the moon's surface or be pulled by the tidal forces toward the planet. Solving for the critical value of r by equating the previous two expressions gives

$$r_{crit} = R_{moon} \left(\frac{2M_{planet}}{M_{moon}} \right)^{1/3}. \tag{5.80}$$

This critical distance is usually referred to as the Roche limit, after the physicist Edouard Roche who first discovered it in 1850.

It is clarifying to re-write this expression in terms of the mass *densities* of the moon and planet, given by

$$\rho = \frac{M}{\frac{4}{3}\pi R^3} \tag{5.81}$$

for each of the two bodies. The result is that the critical radius – the Roche limit – is proportional to the radius of the planet:

$$r_{crit} = R_{planet} \left(\frac{2\rho_{planet}}{\rho_{moon}} \right)^{1/3}. \tag{5.82}$$

For the Earth-Moon system, a quick calculation reveals that the Roche limit occurs at about five and a half times the radius of the Earth – well inside the actual distance to the Moon, about sixty Earth radii. And since the tidal interaction discussed earlier is causing the Moon to slowly *increase* its distance from the Earth, we needn't worry that our (distant) ancestors will someday see the Moon ripped apart.

Interestingly, though, such a fate *does* lie in the future for several other moons in the solar system. Phobos (one of Mars' moons) and Triton (one of Neptune's moons) both orbit their planets in a way that is, in a sense, opposite to the Moon's orbit around Earth. Phobos orbits Mars *faster* than Mars rotates: the Phobosian month on Mars is shorter than a Martian day! In the case of Triton, the moon actually orbits the planet in a retrograde fashion, i.e., opposite the direction of the planet's "daily" rotation. (As seen from way to the north of the solar system, Neptune – like all the other planets – orbits the Sun counter-clockwise, and Neptune – also like all the other planets – *rotates* counter-clockwise. But Triton's orbit around Neptune is clockwise.) In both cases, the effect is to reverse the sense of the slow tidal evolution discussed above for the Earth-Moon system: the two moons in question are (unlike the Earth's moon) getting ever closer to their planets. And so at some point (millions of years in the future) they will reach the relevant Roche limits for their respective planets and be shredded.

Perhaps it has occured to you that the famous rings of Saturn could be the dusty remnants of a tidally shredded moon. Indeed, Saturn does have a number of moons, all of which are farther away than its famous rings. And, indeed, it turns out that (making some reasonable estimates for the density of the moons) the current moons are outside the Roche limit, while the rings are inside. So it is entirely possible that the rings formed, at some point in the past, when tidal (or other) interactions pulled a previously-coherent moon inside the critical radius. Another possibility is that the rings

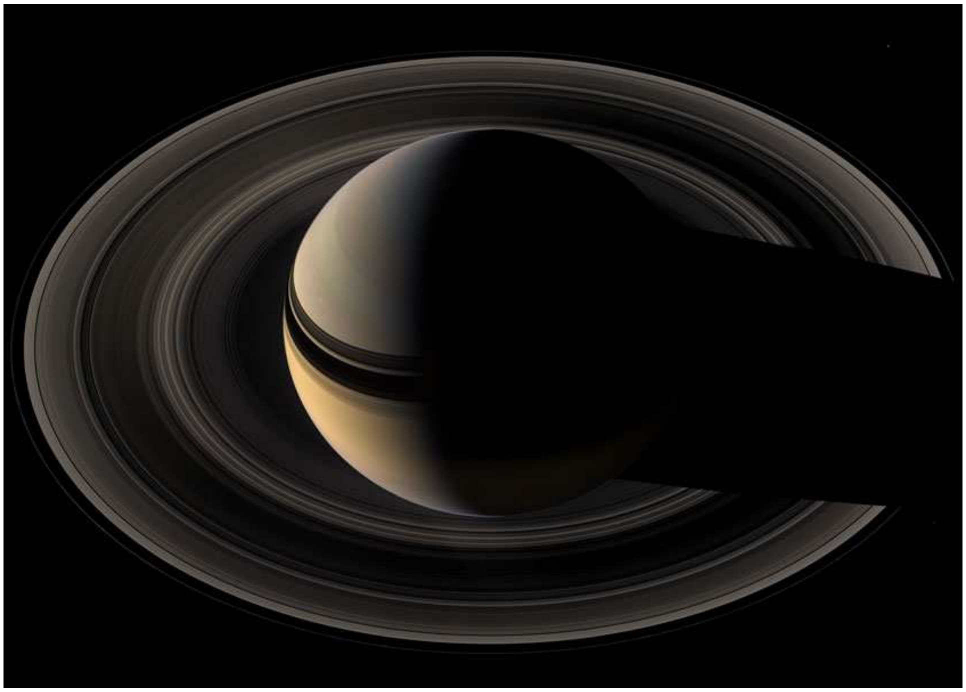

Figure 5.15: A picture of Saturn and its beautiful rings. Several of Saturn's moons are also present, though it is hard to tell that they are all more distant from the planet than the rings. This photo was taken in 2007 by the Cassini spacecraft. What aspect of the picture proves immediately that it wasn't taken from Earth?

are a well-preserved remnant of a primordial swirl of dust that clumped up billions of years ago to form Saturn and its moons. Under this hypothesis, the rings are not the debris of an ex-moon, but rather the ingredients that would have formed a moon had they not found themselves at a distance from the central planet for which the tidal forces prevented the usual moon-formation process of gravitational clumping.

The "tidal cataclysms" we've been discussing here can in principle occur not only for moons which get too close to their planets, but also for planets which get too close to their stars, or even stars in binary systems which get too close to their partners. We will explore this a bit in the Projects. But at least for the case of the moons in the solar system, although the effect is interesting to understand and contemplate, it is a bit moot. For another type of cataclysm will eventually befall many of these systems.

At some point, several billion years in the future, the Sun will start to exhaust the Hydrogen that fuels the internal Hydrogen-to-Helium fusion reactions which power it. As this happens, Helium – the inert by-product of this fusion reaction – will tend to pile up in the core. The inert core will cool somewhat and contract, allowing the still-Hydrogen-rich material above to fall in somewhat, a process which heats the Sun's outer layers. This, in turn, will dramatically increase the rate at which Hydrogen-to-Helium fusion is occuring there, heating the outer layers even further. All this newly generated heat will cause the outer layers of the star to puff up – the net result being that the Sun will become a so-called "red giant" star.

The radius of the newly-formed red giant will exceed the (current) Sun's radius by a factor of about 100 – which means, among other things, that the Sun will then occupy what used to be the orbits of several of the inner planets, probably including Earth. (And even if Earth is spared in this process, the dramatic increase in the Sun's total luminosity will increase the Earth's average temperature far beyond the boiling point of water, making Earth in any case not exactly hospitable.)

Eventually – i.e., after wiping out much of the solar system – the Sun will really run out of Hydrogen fuel, and settle back down to a smaller size. At this point its "life" is essentially over. All that will remain is an inert core of mostly Helium, which will simply sit there and slowly cool off over the subsequent billions of years.

A more dramatic death awaits stars which are significantly more massive than the Sun. For the Sun, the Helium by-product of the primary Hydrogen-to-Helium fusion process is inert – it doesn't participate in any further nuclear processes. But for stars which are ten to a hundred times heavier than the Sun, the temperature and pressure in the star's core greatly exceed those in the core of the Sun. And, it turns out, under such conditions further energy-producing nuclear reactions are possible. For example, three Heliums can fuse together to form Carbon in the so-called triple-alpha process. (The name is because Helium nuclei, which fuse in this process, are also known as "alpha particles.") And likewise, Carbon can fuse with Helium to form Oxygen – which can in turn fuse with another Helium to form Neon – and so on, to heavier and heavier elements.

It is now understood that virtually all of the elements heavier than Helium were created, in stars, in precisely this process. So, for example, the trace amounts of Carbon, Nitrogen, and Oxygen in our Sun (which incidentally act as catalysts in a special Hydrogen-to-Helium fusion reaction called the CNO cycle) signify that the Sun is not a first-generation star, but was rather formed from the remnants of an earlier cycle of stellar evolution. And of course it also means that *we* – who are made of lots of Carbon and Oxygen and Nitrogen – are, in the memorable phrase of Carl Sagan, "starstuff pondering the stars."

Anyway, there is a definite end to this process of fusion reactions leading to heavier and heavier elements in massive stars: Iron. It turns out that fusion reactions from Iron to anything heavier than Iron are *endothermic* – you don't get energy out, but rather have to put it in. That's why nuclear reactors here on Earth – which proceed by *fission*, or the breaking apart of nuclei into smaller components – always begin with elements (such as Uranium) which are heavier than Iron. In terms of nuclear energy, Iron is the bottom of the barrel. You can get energy out by fusing smaller nuclei together or by breaking larger nuclei apart, but once you have Iron you are truly stuck.

So what we said above about Helium in the case of the Sun, applies in a more fundamental, non-negotiable kind of way to Iron for stars massive enough to produce it. That is, the Iron eventually produced by such stars forms a truly inert core, which just grows and grows as the fusion of still-remaining lighter elements continue above it. Since the core is inert, however, it doesn't produce any heat and hence doesn't contribute much to the ability of the star to hold itself up against its own tremendous weight – i.e., against the inherent gravitational attraction of all its parts.

Eventually, the ultimate cataclysm occurs. The inert Iron core is simply unable to support the weight of the material above, and gives way: the entire star implodes, along the way crushing the electrons from the Iron atoms in its core right into their nuclei, where they are literally forced to react with protons. The result is that the core is converted into a uniform and immensely hot soup of neutrons. And now things get really interesting. Eventually, after shrinking in linear size by about 5 orders of magnitude – i.e., after being crushed to something like 10^{-15} times its original volume – the core finally again becomes very stiff, very difficult to compress further. All those neutrons, by virtue of a phenomenon that can only be understood using quantum mechanics, really don't like to get too close together. Over-simplifying only a little, the result is a very big "bounce": the ten or more solar masses worth of material that is racing in toward the collapsing core at tremendous speed suddenly encounter something akin to a brick wall. So all that material bounces off the suddenly solid ball of neutrons that used to be the core, and flies now *outward* at tremendous speed.

The implosion has been converted into an *explosion.* This process is called a core-collapse *supernova.* Most of the material of the star is blown out into surrounding space, often leaving behind an observable remnant called a planetary nebula such as that shown in Figure 5.16.

Also left behind by the supernova explosion is the solid ball of neutrons that used to be the star's inert Iron core. Such an object is called a "neutron star." We've already mentioned that, in the collapse, the volume of the core gets compressed by some 15 orders of magnitude. This is, not surprisingly, about the same as the ratio between the volume of a normal atom and the volume occupied by the atom's nucleus. Thus, a neutron star has roughly the same total mass as the Sun, but an incredibly large density comparable to that (or actually several times bigger than that) of atomic nuclei. A single teaspoon of neutron star material would weigh as much as a billion cars! Perhaps more dramatically, this means that the neutron star has a radius of only about 10 kilometers. A neutron star thus has as much matter as the Sun, compressed into a ball no larger than a small town!

Neutron stars don't shine in the visible part of the spectrum the way normal stars do, but they can be detected and observed by astronomers nevertheless. The first observation of (what was only later identified as) a neutron star occured in 1967 when two radio astronomers, Jocelyn Bell and Antony Hewish noticed a curious and extremely regular pulsation in the radio signal coming from a certain direction in the sky. They initially thought the signal must be some kind of noise in the apparatus, or of some other terrestrial origin, because the precisely-regular beep-beep-beeping seemed too strange to admit a heavenly origin. But that conclusion eventually became inescapable, and the mysterious astronomical source was dubbed a "pulsar."

The discoverers briefly considered the possibility that the beeping was being emitted by extra-terrestrials! But cooler heads prevailed, and in time the consensus developed that pulsars were rotating neutron stars, emitting a burst of radio-wave radiation toward us each time a certain part of their magnetized bodies passed by.

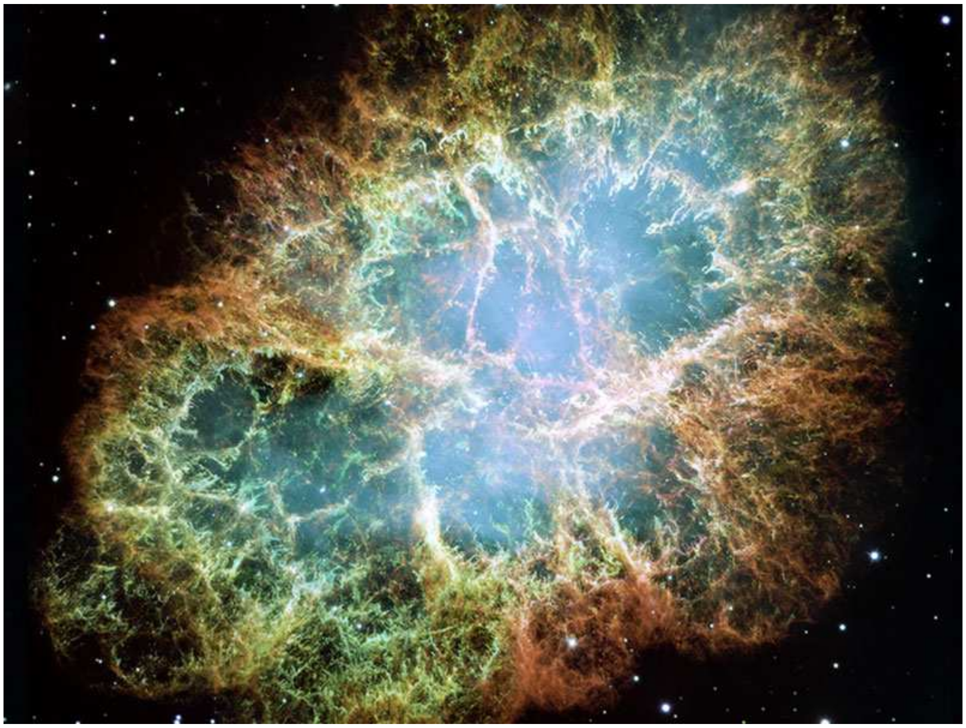

Figure 5.16: The Crab nebula. The supernova which produced it was visible to the naked eye from Earth and was actually observed and recorded by Chinese astronomers in 1054. Buried in the rubble is a rotating neutron star – the Crab pulsar – which was first identified by radio astronomers (but not yet understood to be a rotating neutron star) in 1968.

Many hundreds of pulsars were subsequently discovered, most with rotational periods of around one second. Note that this is one of the key pieces of evidence in favor of interpreting puslars as rotating *neutron* stars: if an ordinary Sun-like star were rotating with a period of one second, the result would be not just a small Equatorial bulge, but complete centrifugal annilihation (like the batter on the electric mixer pulled too soon out of the bowl).

Actually, even some neutron stars are fairly close to this limit. The Crab pulsar – the rotating neutron star in the Crab nebula shown in Figure 5.16 – has a period of only 0.033 seconds. And other pulsars have been discovered whose periods are as short as a few milliseconds. But no pulsars have been observed with periods less than a millisecond. This is additional evidence for the rotating neutron star model of pulsars. Following the calculation above of the critical distance for tidal disintegration, we may estimate the critical period for centrifugal disintegration as follows.

Assuming a roughly spherical body of mass M and radius R, the gravitational field at the surface has magnitude

$$g = \frac{GM}{R^2} \tag{5.83}$$

while the centripetal acceleration of a point on the Equator (i.e., the centrifugal contribution to the effective gravitational field there, if we use a co-rotating reference frame) is

$$g_c = \omega^2 R = \frac{4\pi^2 R}{T^2}. \tag{5.84}$$

If these are equal, it means the disintegrative centrifugal force (on, say, some random neutron near the Equator) is comparable to the gravitational force keeping it together with the rest of the star. We may thus set them equal and solve for the critical rotation period

$$T_{crit} = 2\pi\sqrt{\frac{R^3}{GM}}. \tag{5.85}$$

If the rotation period is shorter than this (i.e., if the rotation is faster), the body will be torn apart by centrifugal forces.

What is the critical rotation period for a neutron star? Plugging in the rough numbers $M \approx M_{sun}$ and $R \approx 10\,km$ gives

$$T^{NS}_{crit} \approx 0.5\,ms. \tag{5.86}$$

So, on the premise that pulsars are rotating neutron stars, we can understand why no sub-millisecond pulsars have been observed.

The intriguing question of how a neutron star could get to be rotating up to a thousand times per second will be left for the Projects.

5.6 New Discoveries

Not only can gravitation be used to indirectly measure masses of previously known objects like moons and stars – it can also be used to discover entirely new objects! A contemporary terrestrial example of this was noted earlier in the chapter: sensitive modern instruments can measure the gravitational field $\vec{g}$ near the surface of the Earth with such great precision, that the tiny local fluctuations produced by, for example, underground mineral deposits can be detected. Such measurements have thus allowed scientists to know, beforehand, where to dig or drill to tap into valuable natural resources. This is a remarkable and beautiful example of the long-term practical benefits of progress in basic science.

5.6.1 New Planets

A less immediately practical but even more dramatic example of using gravitation to make new discoveries occurred in the 19th century. Recall that, according to the Ancient Greeks, there were (in addition to the Sun and Moon) *five* planets: Mercury, Venus, Mars, Jupiter, and Saturn. Of course, with the Copernican revolution, it was realized that the

Earth too was a planet, bringing the number to six. In the last section we noted the 20th Century discovery of the ninth planet (or, at any rate, what was formerly considered the ninth planet), Pluto. But when and how were the seventh and eighth planets – Uranus and Neptune – discovered?

Uranus was first recognized as something other than an ordinary star by the great English astronomer William Herschel in 1781. He stumbled on it essentially at random, in the course of his ongoing systematic surveys of the heavens. Herschel originally suspected that the newly discovered object was a previously unobserved comet, but subsequent observations revealed a more-or-less circular orbit around the Sun with a period of about 84 years. This object, subsequently named Uranus, was therefore a new planet. Its orbital radius was about 19 AU, or roughly twice that of Saturn, which previously marked the outer fringe of the known solar system.

Over the subsequent decades, though, the increasingly detailed observations of Uranus' actual motion increasingly failed to match up with theoretical expectations. This is not to say, for example, that Uranus orbited the Sun in a square rather than an ellipse, in gross violation of Kepler's laws. Actually, by this time it was known that all of the planets violated Kepler's laws to some small extent, because their orbits are influenced not only by the gravitational force of the Sun, but also by small gravitational forces exerted by the other planets. The point here is that the observed motion of Uranus seemed anomalous *even when these tiny inter-planetary perturbations were taken into account.*

Two quite reasonable hypotheses arose to explain the discrepancy. One possibility was that Newton's theory of gravitation simply didn't apply for an object at such a tremendous distance from the Sun. After all, Kepler's laws – taken here as summaries of the motion of the planets known about by Kepler – were the central pieces of evidence for Newton's theory, and that evidence pertained only to objects whose separation was at most the distance between Saturn and the Sun. There simply was no direct empirical data to support the extrapolation of Newton's inverse square law to longer distance scales. And clearly, by *some* appropriate modification to Newton's formula (i.e., by inventing the right fudge factor) the anomalous behavior of Uranus could be accounted for.

The alternative hypothesis was the existence of *another* previously-unknown object, whose gravitational influence on Uranus could in principle account for the small anomalies in its observed motion. This idea remained just another speculative gesture toward an appropriate fudge factor until two scientists, John Adams from England and Urban Leverrier from France, undertook to calculate the precise position and orbit of the hypothesized object. Adams and Leverrier worked independently and didn't know of each other's work, and the successful outcome led to a great international controversy. Adams probably finished the relevant calculations first, but his request to astronomers at the English Royal Observatory went unheeded for some time, since Adams was "merely" an unknown mathematician.

Leverrier, on the other hand, sent his predictions to a colleague at an observatory in Berlin. The eighth planet, Neptune, was discovered right away, in 1846, in just the region of sky that Leverrier (and Adams) had predicted. Neptune had an orbital radius of about 30 AU, and an orbital period of 165 years.

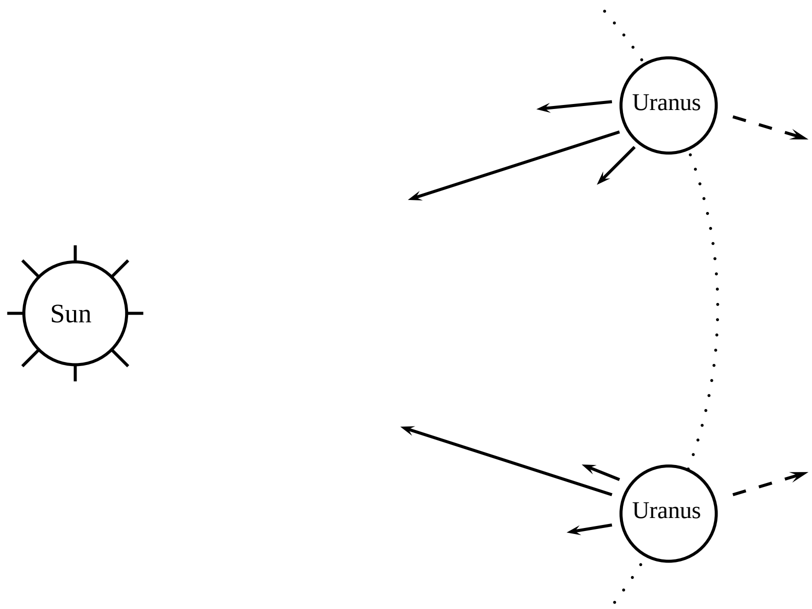

Figure 5.17: Schematic description of the calculations made by Adams and Leverrier. The acceleration of Uranus is produced by the joint effect of the gravitational forces exerted by the other bodies in the solar system. The forces exerted by the Sun, Jupiter, and Saturn are indicated by the three solid arrows, at two different times. Since the actual acceleration can be inferred from observation, the "anomalous forces" – the dotted-arrows in the Figure – can be computed. These are the gravitational forces exerted by the hypothetical new object, which of course turns out to be the new planet Neptune. Note that if the new planet were stationary, its position could be calculated by "triangulation." But since it, too, is expected to be in orbit around the Sun, the calculation is a little more subtle. If its own orbital radius were known, Kepler's third law would tell us the rate of its motion around the Sun, and its motion could be explicitly corrected for in the triangulation. In fact, both Adams and Leverrier made what turns out to have been a rather bogus assumption about the orbital radius of the hypothetical new object, based on a curious (one might say numerological) regularity in the orbital radii of the then-known planets called Bode's Law. As it turns out, their assumption about the orbital radius of Neptune was off by about 20%. It was a matter of sheer dumb luck that this bogus assumption didn't significantly affect Adams' and Leverrier's predictions!

Figure 5.17 gives a somewhat schematic indication of the kinds of calculations Adams and Leverrier made to predict the existence of Neptune, whose successful discovery was regarded as a major triumph for Newton's theory of gravitation. One scientist later described it this way:

> "The explanation by Newton of the observed facts of the motions of the moon, the way he accounted for precession and nutation and for the tides, the way in which Laplace [using Newton's theory] explained every detail of the planetary motions – these achievements may seem to the professional astronomer equally, if not more, striking and wonderful.... But to predict in the solitude of the study, with no weapons other than pen, ink, and paper, an unknown and enormously distant world, to calculate its orbit when as yet it had never been seen, and to be able to say to a practical astronomer, 'Point your telescope in such a direction at such a time, and you will see a new planet hitherto unknown to man' – this must always appeal to the imagination with dramatic intensity."

Actually, the same story more or less repeated itself, in a slightly less dramatic way, with the discovery of Pluto. In the decades after Neptune's discovery, *its* orbit was observed to deviate slightly from theoretical predictions, just as had that of Uranus decades earlier. This time, however, the deviations were much smaller. And so, although people this time guessed right away that the deviations were probably caused by yet another previously unknown planet, it was much harder to get a reliable estimate of that undiscovered planet's location. Pluto was finally discovered in 1930 as a result of these calculations, but this was after several decades of failed searches. And the specific calculations which led directly to Pluto's discovery were subsequently shown to be erroneous (in a more significant way than were those of Adams and Leverrier). Persistance and dumb luck thus played a great enough role in Pluto's discovery that it usually isn't considered any great triumph of Newton's theory of gravity. Nevertheless, it was ultimately Newton's theory which made that discovery possible, if only in an indirect sense.

We now understand better why the search for Pluto was so fraught with difficulty. Pluto is significantly less massive than any of the other planets – the next smallest, Mercury, is 20 times heavier! Thus, Pluto's gravitational perturbation on Neptune is very small. Moreover, Pluto turns out to be just one of a larger group of small, planet-ish objects occupying the outer fringes of the Solar System and marginally perturbing the orbit of (especially) Neptune. As Pluto was the first of these so-called trans-Neptunian objects (TNOs) to be discovered, it was naturally treated initially as another planet. But as more and more TNOs were discovered in the 1990s and 2000s, it became increasingly clear that Pluto had more in common (including its size, composition, and orbital character) with these other objects than it did with the eight planets. Pluto turns out not to even be the biggest of the TNOs. So you can see why Pluto was recently demoted from full planetary status – i.e., using a neologism inspired by this controversial episode, why Pluto was "plutoed."

5.6.2 Exo-planets

In the 1990s, the discovery of new planets extended beyond our own solar system for the first time. Of course, once it was understood that the stars were more or less like the Sun, only farther away, it became natural to speculate that other stars, like the Sun, would be centers of planetary systems. Newton even mentions this possibility in the Principia. But the first genuine empirical discovery of a planet orbiting a Sun-like star was made only very recently, in the 1990s. The star in question is 51 Pegasi, and its planet – *51 Pegasi b* – was detected indirectly, via its gravitational influence on the star.

The physics here is very similar to that presented already in the discussion of binary systems such as Pluto and Charon or binary stars. The idea is that, strictly speaking, the planet doesn't orbit a central, fixed, star. Rather, the star and planet both orbit around their mutual center of mass. Since only the star is directly observable (at least with current technology, and even this is starting to have exceptions), the planet manifests itself in the tiny back-and-forth periodic wiggle of the position of the star. Observations of the amplitude and period of this wiggle then allow some inferences about the properties of (or, less specifically but more profoundly, the *existence* of) the planet.

Actually, just as with the mass determinations of binary stars, it is more common (i.e., currently possible/easier!) to observe not the back-and-forth wiggle in space, but, instead, the back-and-forth fluctuations in *frequency* from which the periodic oscillations in the radial velocity can be inferred. Then the same formalism we developed before – Equations 5.75 and 5.76 – can be used to infer the mass and orbital radius of the invisible planet.

This is precisely the method that astronomers used to discover *51 Pegasi b*. A graph of the radial velocity of the parent star, 51 Pegasi, as a function of time is shown in Figure 5.18, and shown again as a function of the phase of the inferred periodic cycle, in Figure 5.19. The star moves in and out relative to us with a period $T = 4.23\,days$ and an amplitude of about $56\,m/s$. Unfortunately, the planet does not appear to pass in front of the star during its orbit, so the inclination i of its orbit remains unknown. Nevertheless, it is possible to put a lower limit on the planet's mass. This turns out to be

$$m_2 \geq .45\,M_{Jup} \tag{5.87}$$

where M_{Jup} is the mass of (our own) Jupiter. It is also possible to infer from the data that 51 Pegasi b has an orbital radius of $R_2 = .05\,AU$. So the planet is (probably) roughly as big as Jupiter, but – compared to the real Jupiter – very close to its parent star. It, and the many other extra-solar planets like it which have been subsequently discovered, are therefore sometimes called "hot Jupiters."

You should probably be wondering: how exactly did the scientists determine the *mass* of this extra-solar planet? In our discussion of measuring masses in binary systems (such as the Pluto-Charon system or a double star system), we found that one must determine empirically not only the period of the orbit(s), but also the radii or maximum radial velocities of *each* of the two bodies, in order to determine either of the masses. Recall, for example, Equations 5.75 and 5.76. But the extra-solar planet discussed here remained *invisible*: so while the period and v^{max} of the *star's* wobble could be observed,

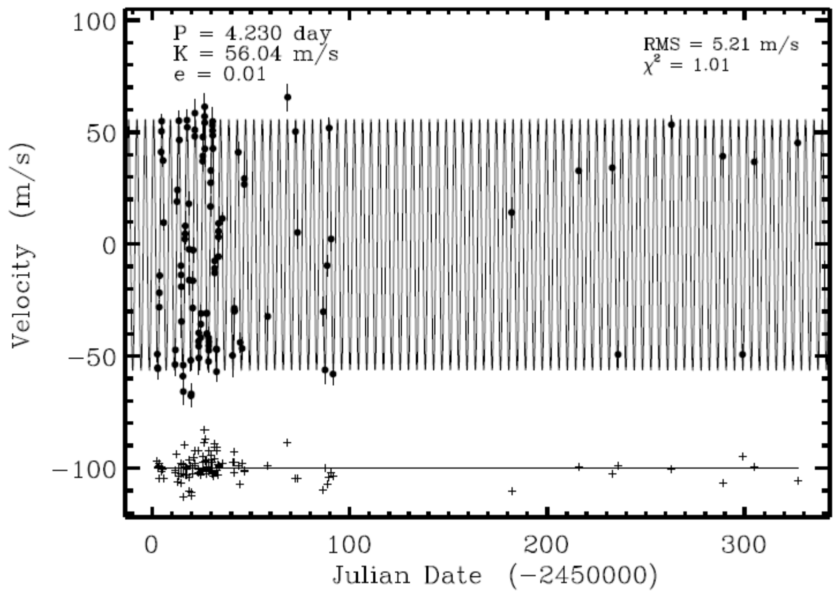

Figure 5.18: Data for the radial velocity (inferred from Doppler effect observations) of the star 51 Pegasi over the course of about a year. The wiggly line is a sinusoidal fit to the data, which maybe looks a little suspicious given the seemingly random character of the data. The residuals for the fit, however, are shown below and indicate that the fit is quite good. (Observations of nearby, non-wiggling stars indicate that there is about a 5 m/s uncertainty on any of the velocity measurements – so the residuals are just the size one would expect given the inherent accuracy of the data.) Note also how the data were taken over the course of the year: lots and lots of observations over a month or two to get an accurate guess of the periodicity, and then just a few measurements, almost randomly spaced over the subsequent months, to test whether the guessed periodicity continues to fit the data over a longer timescale. As the points to the right in the plot of the residuals shows, it does. This provides much more confidence that the fit is correct, than would (say) the same total number of data points crammed into just a month of observation, or the same number of data points uniformly spaced over an entire year.

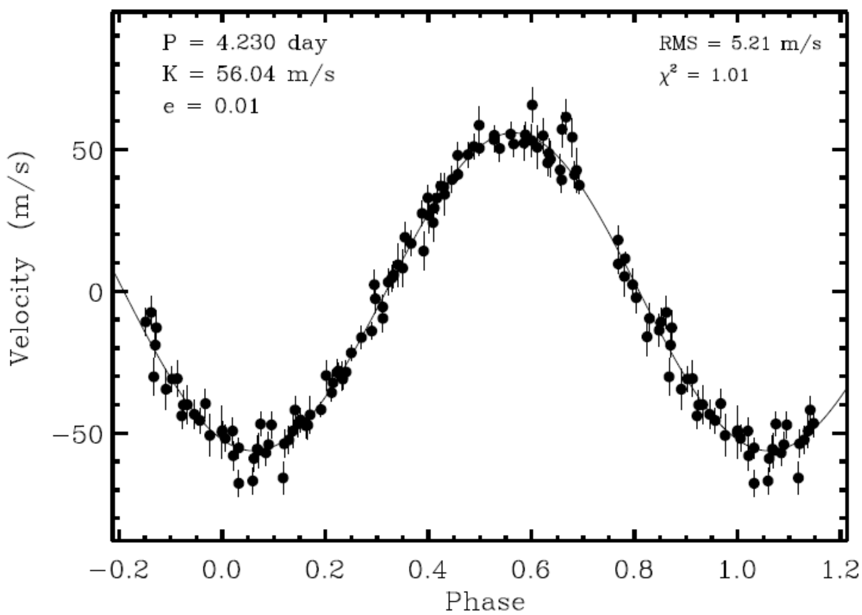

Figure 5.19: The same data as the previous figure, but plotted as a function of the phase of the inferred 4.231 day cycle. This makes the sinusoidal character of the in-and-out motion of the star particularly clear.

the v^{max} for the actual planet could *not* be. One of the three crucial pieces of evidence seems to be missing!

Here is the resolution to this puzzle: we can make an educated guess about the mass of the star by measuring its luminosity and then using the mass-luminosity relation discussed above and shown in Figure 5.14. This, of course, requires an assumption that the star in question is relevantly like the stars whose masses and luminosities were shown to be so correlated. But there is abundant evidence for this hidden away in the light emitted by the stars – in particular, in their *spectra*, i.e., in the distribution of their emitted light across the frequency spectrum.

Most of the subsequently discovered extra-solar planets were discovered by more or less the same method. And so that's, in a nutshell, how scientists in recent decades have established that, as had long been suspected, there do exist planets orbiting stars other than our own (the Sun) – and how they measure the mass and orbit of the planets to boot. At this writing, hundreds of extra-solar planets have been positively detected, and the rate of their discovery is continuing to accelerate.

5.6.3 Dark Matter

Let us close with one more example of a recent discovery made using Newton's theory of gravitation: the discovery of so-called "dark matter." This follows roughly the same pattern discussed above under the heading of measuring masses. Moons orbit planets and planets orbit stars, and the orbital character of the orbiting body (in particular its period and radius) can be used to infer the mass of the central gravitating body. Similarly, it turns out that stars arrange themselves in enormous clusters called galaxies, with the individual stars all (more or less, on average) orbiting around the galactic center.

A particularly beautiful type of galaxy – see for example Figure 5.20 – has most of the stars clumped up into a spiralling disc.

Using the same Doppler-effect-related spectroscopic techniques described before, astronomers can measure the speed with which individual stars (or groups of them) orbit around the center of their galaxies. For a star on the outer fringes of its galaxy, the orbital speed should be given approximately by the familiar Newtonian calculation which sets the centripetal gravitational force (produced collectively by all the stars in the galaxy) equal to the mass m of the star in question times its centripetal acceleration, $a_c = v^2/R$. Thus we expect

$$\frac{GMm}{R^2} = m\frac{v^2}{R} \tag{5.88}$$

where M is the total mass of the galaxy and R is the galactic radius of the star in question. This reduces to

$$v = \sqrt{\frac{GM}{R}}. \tag{5.89}$$

Of course, we've assumed here that the rest of the galaxy can be treated as if it were a single point of mass M located at the galactic center. For a star anywhere near the middle of the galaxy, this is a terrible approximation – for such a star, "the rest of the galaxy" will be pulling it in several directions at once and from several different distances. But for stars out on the outer fringes of the galaxy, "the rest of the galaxy" *is* all pulling it in the same direction. Of course, the mass is distributed in something like a disc shape (not a perfect sphere) so we might worry that there are corrections to the simple point mass formula like those we dealt with in discussing the Earth's Equatorial bulge. And, indeed, such corrections should exist. Nevertheless, they will be increasingly small corrections for stars that are truly on the fringes, very far from the galactic center.

All of this is just meant to underscore that, although Equation 5.89 is derived with the crudest possible approximations, we have good reason to think it should apply to stars on the outer fringes of galaxies. Yet, when the orbital velocities of such stars are actually *measured*, they do not appear to vary with R in the way that Equation 5.89 suggests they should – i.e., decrease with R as $1/\sqrt{R}$. Instead, what is observed is that the orbital velocities of stars on the outer fringes of galaxies tend to be quite *constant* – independent of R. See Figure 5.21.

What does this mean? Obviously it means that one of the assumptions we've made in generating the wrong expectation, is itself wrong. One possibility (again, just like in the discussion of Neptune's discovery) is that Newton's formula for the gravitational

Figure 5.20: The galaxy M51, also known as the Whirlpool Galaxy. It is located about 30 million light years away, and has a radius of (roughly, since there's no well-defined edge) about 30 thousand light years or about 9 kiloparsecs. (Recall that a parsec is the distance a star would have to be from the Sun in order to exhibit a parallax of one second of arc, i.e., 1/3600 of a degree. The closest stars to the Sun are about a parsec away, which is about 200,000 AU. It's nice to have a sense of the relative order of magnitude of these things. To summarize: the nearest stars are hundreds of thousands of times (5 orders of magnitude) farther away from us than the Sun. And the galaxy – ours turns out to be roughly the same size as the Whirlpool – is another 10,000 times (four orders of magnitude) bigger than that. The galaxy, then is some 9 orders of magnitude – a billion times – bigger than the Earth's orbit around the Sun. The distance between Galaxies is then another factor of a thousand – three more orders of magnitude – bigger than that. And it turns out galaxies themselves form clusters, with relative gaps between *them*. And believe it or not, even the galaxy clusters form clusters – "superclusters" they're called. So there is important and interesting structure in the universe across an incredibly broad spectrum of length scales. And we haven't yet even begun to discuss the *small* end of the spectrum!

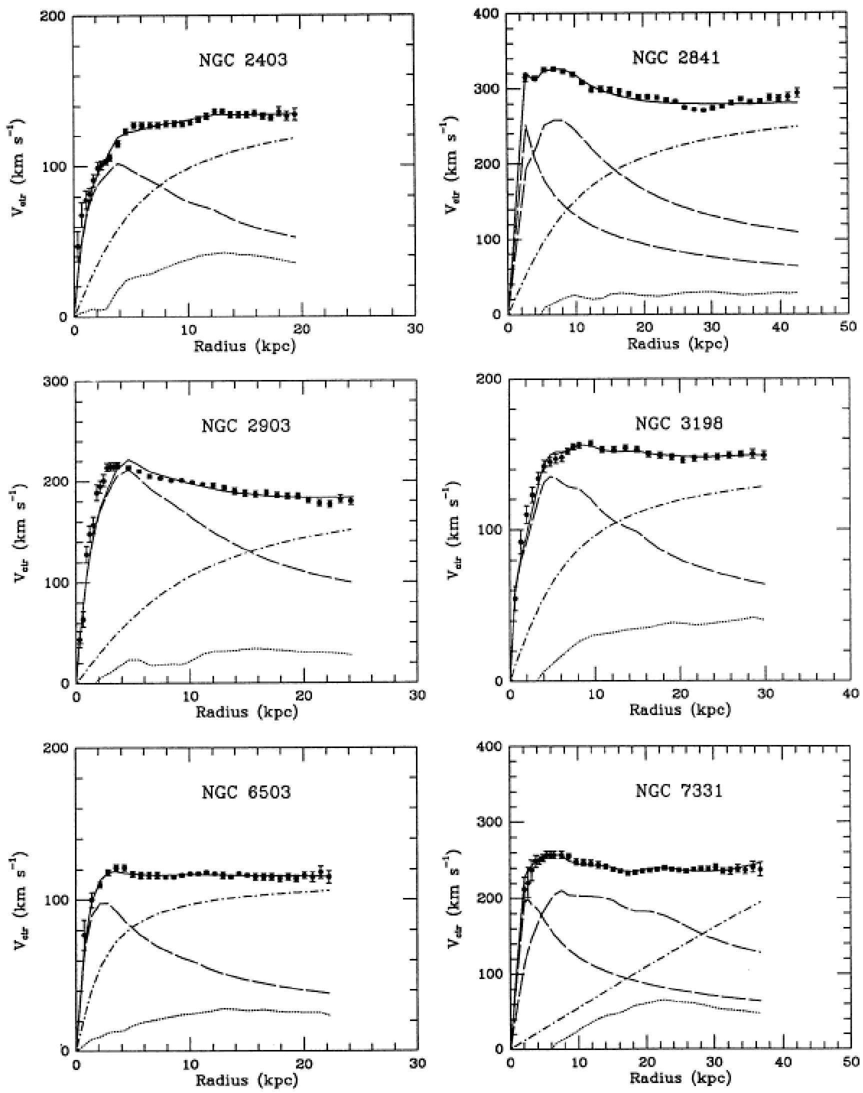

Figure 5.21: Observed rotation curves for six "typical" galaxies. Dots are data points for the rotational velocity (as measured via the Doppler effect). The three curves below are the components of a three-parameter fit to the rotation curve data: "the dashed curves are for the visible components, the dotted curves for the gas, and the dash-dot curves for the dark [matter] halo. The fitting parameters are the mass-to-light ratio of the disc (M/L), the halo core radius (r_c), and the halo asymptotic circular velocity (V_h)." Image and parts of caption from "Extended rotation curves of spiral galaxies – Dark haloes and modified dynamics" by K. G. Begeman, A. H. Broeils, and R.H. Sanders, Royal Astronomical Society, Monthly Notices, vol. 249, April 1 1991, pages 523-537.

force simply doesn't apply at these (now *really*) large distance scales. This is considered a going hypothesis in current research – the idea generally goes by the name Modified Newtonian Dynamics, or MOND for short.

But by far the more popular interpretation of the surprising data is the hypothesis of so-called "dark matter." The idea is that, although the stars we have been talking about *appear* to be on the fringes of the galaxy, in the sense that virtually all of the *observable* matter (i.e., the other stars) are much nearer the galaxy's center, in fact those stars are not near the galaxy's "edge" because the galaxy consists not just of the visible stars but also of some mysterious non-visible ("dark") matter which, nevertheless, gravitates.

Another way to put the problem and the (hypothesized) solution is this: if you just calculate, using Equation 5.89 and the actual velocities and radii for some stars near the (apparent) fringes of a galaxy, the mass M of the galaxy – that calculated mass is substantially *bigger* than the mass you would have guessed by counting up all the stars in the galaxy and multiplying by the average mass of a star. So there must exist, in addition to the stars (which both gravitate and produce light), some "dark matter" (which gravitates but does not produce light).

Note that the dark matter is no mere marginal correction. Current estimates (based not only on the velocities of stars in galaxies, but several other methods as well) suggest that there is something like five or ten times more mass in dark matter than in ordinary "light matter" (mostly stars).

All of this obviously raises the question: what *is* this dark matter? No answer can be given, because it is simply not yet known. Some have speculated that the dark matter is ordinary matter that does not produce light – e.g., billions of Jupiter-sized "planets" roaming around the universe. This is an intriguing possibility if only because it doesn't require the postulation of any wholly new type of matter. It is, however, very difficult to understand where all these Jupiters would have come from. (Suffice it to say that otherwise strongly-confirmed theories of the evolution of stars and planets do not suggest that such Jupiters could be produced in the needed numbers and with the needed spatial distribution.) Other proposed dark matter candidates include exotic new sub-atomic particles (beyond the electrons and quarks of which ordinary matter is made). Such models are, in a way, more consistent with the overall astronomical evidence. But they suffer from the fact that none of the candidate particles have ever been observed in particle physics experiments.

The identity of dark matter thus remains a profound mystery.

And while it may perhaps feel a bit anti-climactic, that is a fitting way to close our survey of the astrophysical applications and implications of Newton's theory of gravitation. As we have seen, Newton's theory forms the crucial support for virtually everything we have discovered about our world and our universe. But it also continues to provide the basic context for the questions and puzzles at the current frontier of our knowledge. Surely there could be no stronger testament to the theory, especially considering that we are now well into the fourth century after its publication!

Questions for Thought and Discussion:

1. Suppose a piece of pizza dough were thrown up, spinning, repeatedly. (Or equivalently: suppose it were set spinning in outer space.) Would it keep getting flatter and flatter indefinitely? Or would it, like the Earth, reach some kind of equilibrium beyond which further flattening would increase the total energy? What's the relevant difference, if any, between the pizza dough and the Earth?

2. Everyone knows that the highest point on Earth is the top of Mt. Everest (on the border between Nepal and Tibet). But actually, this depends on what you mean by "highest." The point on Earth *whose distance from the Earth's center* is greatest, is the top of Mt. Chimborazo in Ecuador. What is going on here? How can the highest point (by the usual meaning of "highest") not be furthest from the center? What exactly is the "usual meaning" of "highest"?

3. Because of the bulging of the earth near the equator, the source of the Mississippi River, although high above sea level, is nearer to the center of the Earth than is its mouth. How can the river flow 'uphill'?

4. If you turn your car to the right, you experience being pulled to the left, e.g., pushed up against the left side of the interior of the car. Is there really a force pushing you left?

5. Suppose the Earth were perfectly spherical. Would your weight change as a result of moving North or South, i.e., changing your latitude? Would the reading of your bathroom scale change? Explain. What does a bathroom scale actually measure?

6. Near the equator, during what part (or parts) of the year are high tides highest and low tides lowest? How about near the poles? How about in middle latitudes?

7. The caption to Figure 5.7 perhaps gave the impression that – wherever there *are* two low tides per day – the two low tides should be equally low. As you may have noticed on ocean visits, this is not true. The two low tides each day are not necessarily equally low. The question is: can this be understood from the "equilibrium" model of tides that most of the text's discussion (and Figure 5.7 in particular) is based on? Or must we resort to the complicated sloshing of tidal waves to understand this? To make the assignment a little more concrete: can you come up with a scenario (i.e., a relative arrangement of the Earth, Moon, and Sun) in which (say) observers at middle latitudes will experience two low tides that are not equally low?

8. Would you expect the amplitude of tides to be higher in Hawaii, or in Florida (at about the same latitude)? Why?

9. How, if at all, would the tides be different if, instead of orbiting one another, the Earth was rigidly stapled to the cosmic graph paper, with the Moon orbiting around it?

10. When we calculated the height h of the (rotation-produced) Equatorial bulge, our formula was too small by roughly a factor of two because we ignored the gravitational effect of the Equatorial bulge itself. (Thinking in terms of $\vec{g}_{eff}$ near the surface of the Earth, the point is that the Equatorial bulge makes the true gravitational acceleration $\vec{g}$ itself have a "true horizontal" component. Or, thinking in terms of the energy argument, the point is that a mass of material would actually be moving gravitationally "downhill" in going from the pole to the Equator, i.e., $\Delta PE^{A \to B}$ is not zero, but negative and about half as big as $\Delta PE^{B \to C} = mgh$.) Was there a parallel error in our calculation of the height h of the tides? That is, is the true equilibrium tide height h (the altitude difference between the high and low tide points) produced by the Moon about twice what we said? About a meter, rather than $54\,cm$?

11. The text discussed how, in millions of years, the Earth and Moon will become "tidally locked" in a face-to-face dance in which the same face of the rotating Earth is always pointing toward the Moon. It was mentioned in passing that the Moon *already* orbits in such a way that it presents the same constant face to the Earth. Why do you think it does this? Is this just a coincidence? (You better not say yes – it is extremely common for moons orbiting other planets in the solar system to orbit this way!) If your answer has something to do with tides, does this square with the fact that the Moon is dry (no oceans)?

12. Following up on the previous question, can you explain why it is so common for moons in the solar system to have very circular (as opposed to highly elliptical) orbits?

13. The text encouraged you to think about the gradual increase in the orbital radius of the Moon in terms of the Work-Energy theorem. But what was said there was actually a little sketchy. It's true that the net gravitational force exerted by the Earth on the Moon has (because of especially the near-side tidal bulge) a small "easterly" component. We hinted (too quickly) above that this was a component of the net force that was parallel to the direction of motion. Hence, by the Work-Energy theorem, we said, positive work is being done on the Moon and so its total energy should increase – which we then interpreted as meaning that its orbital radius should increase. But that's not what the Work-Energy theorem says! The theorem says that the net work done on an object should equal the change in its *kinetic* energy. But when the orbital radius of an orbiting body (in a roughly-circular orbit) increases, its kinetic energy does *down*, not up! (You should prove this to yourself.) Does this mean the Work-Energy Theorem is actually contradicted by the behavior of the Earth-Moon system? How can you resolve this paradox?

14. Consider the distant future in which the Earth-Moon system has become tidally locked. Now step back and think about the system comprising the Sun and the Earth-Moon. What should happen in the even more distant future?

15. Do you think Pluto should be classified as a planet? Why or why not? What, if anything, hinges on this question? Is it a pointless discussion?

16. Our discussion of determining the masses of the stars in binary star systems assumed that the stars' orbits were *circular*. Is this necessary? Is there any reason that highly-elliptical orbits for stars in binary star systems should be rare?

17. We all live well inside the Earth's Roche limit. How come we aren't ripped apart by tidal forces?

18. Think about the calculations that Adams and Leverrier made to predict the location of Neptune, as sketched in Figure 5.17. Can you understand why they needed to make some assumption about Neptune's oribtal radius? Strictly speaking, given Kepler's thrid law, wouldn't only one orbital radius be consistent with the two "anomalous forces" shown in the Figure? So why was this assumption necessary? Think about what other factors were glossed over in the text, including the precision with which these anomalous forces could be calculated. Also, roughly what period of time must elapse between the two times when Uranus' position (and the forces acting on it) are shown in that figure?

19. The extra-solar planet discussed in detail in the text, 51 Pegasi b, was described as a "hot Jupiter." Though not all of the other currently known extra-solar planets are hot Jupiters, *most of them are*. Do you think this means that most planets outside our own solar system are hot Jupiters? Why or why not?

20. The text explained how, by measuring the masses of stars in binary systems close enough to the earth that their intrinsic luminosities can also be calculated, the empirical Mass-Luminosity relation (plotted in Figure 5.14) was worked out. Explain qualitatively how, once this Mass-Luminosity relationship is known, one could use it to determine the distance to another binary system which, say, is sufficiently far away that its distance cannot be determined by parallax.

21. An ordinary star is in a close binary orbit with a neutron star. Suppose now that the ordinary star becomes a red giant, such that its outer surface gets inside the neutron star companion's Roche lobe. What will start happening, and what do you think will happen to the neutron star eventually?

Projects:

5.1 In the text, we derived Equation 5.13 for the oblateness of a rotating sphere like the Earth in two different ways. There is a third way, which is probably easier than the other two (especially once you understand the other two!). It involves using energy considerations as in the first method, but using a non-inertial co-rotating frame of reference as in the second method. The crucial point is that the centrifugal force which appears in the co-rotating frame implies an additional contribution to the potential energy. Work out an expression for this, and use it to re-derive (yet again) Equation 5.13.

5.2 Let's try to estimate the quantity $\Delta PE^{A\to B}$ that plays an important role in the calculation of how the Earth's oblateness depends on its rotation rate ω. The simplest model is probably to think of the Earth as a perfect sphere plus a "hula hoop" near the Equator. The spherical part is, of course, spherically symmetric and so won't contribute anything to $\Delta PE^{A\to B}$. We need then only try to estimate the contribution from the "hula hoop." To begin with, write down some approximate expressions for the mass and radius of the hula hoop, in terms of the total mass of the Earth M, its radius R, and the height of the Equatorial bulge h. (The hoop's mass should probably be something like the total mass of the Earth times the fraction of the Earth that is in the Equatorial bulge as opposed to the underlying spherical core, which fraction will have to be estimated. The hoop's radius should probably be R_{earth} or $R_{earth}/2$ or something like that, to take account of the fact that not all of the Equatorial bulge is right at the Equator, i.e., much of the mass of the bulge is closer to the spin axis than R_{earth}.) Now use calculus to develop expressions for the potential energy of a point mass m a distance r from the center of a hula hoop (radius R_{hoop} and mass M_{hoop}) (a) along the symmetry axis and (b) in the plane of the hoop. (For this problem it will be sufficient to expand these expressions in powers of R_{hoop}/r and drop terms smaller than R^2_{hoop}/r^3, if you want.) Now compare the potential energy at the same distance, $r = R_{earth}$, along the two different directions – i.e., at the Pole vs. at the Equator. You should find that the difference in potential energy is

$$\Delta PE = -\frac{3}{4}\frac{GM_{hoop}mR^2_{hoop}}{R^3_{earth}} \tag{5.90}$$

which should reduce to something in the neighborhood of

$$\Delta PE \sim -\frac{1}{2}mgh \tag{5.91}$$

but with probably, some other dimensionless fraction (like 2/3 or 9/32 or something) out front, depending on exactly what you said when you estimated M_{hoop} and R_{hoop}. So the point of this calculation is only to show that you can get in the general ballpark of the result claimed in the text – namely, that $\Delta PE^{A\to B}$ is *in the neighborhood of* $-1/2$ times $\Delta PE^{B\to C} = mgh$, which effectively *doubles* the prediction for the Earth's Equatorial flattening, bringing that prediction very very close to the actual, empirically measured value.

5.3 Here's another nice model for the not-quite-spherical Earth. This has the advantage of being simpler than the sphere-plus-hula-hoop model considered previously, but the disadvantage of failing to possess the same rotational symmetry as the actual Equatorially bulging Earth. That can cause problems, but is actually OK so long as we restrict ourselves to discussing features of the Earth's gravitational field that are confined to some cross-sectional plane like that shown in Figure 5.1. Here is the model: pretend that the Earth is a dumbbell, i.e., two point masses separated by some distance and both located near (but not quite at) the real Earth's center. Suppose that the two masses each have one half of the Earth's total mass so that, together, they are (in terms of total mass) equivalent to the actual Earth. Then: what should their separation be in order to reproduce the empirical fact reported at the end of section 5.1 – namely, that at equal distances R_{earth} from the center, the gravitational acceleration near the Pole is $0.048\,\mathrm{m/s^2}$ smaller than the gravitational acceleration near the Equator? The idea here is to take this empirical fact as fixing the (otherwise free) separation parameter in the model. We can then test the accuracy of the model, for example, as follows: what does it predict for the quantity $\Delta PE_{A\to B}$ that plays some role in the energy-based calculation of the size h of the Equatorial bulge? (Note: later, in Project 5.8, we will use this same model to calculate the period of the Earth's precessional motion!)

5.4 Jupiter is an oblate spheroid just like Earth, but with an observationally measured flattening parameter of $f \approx .065$. This flattening is so large that Jupiter's oblate shape is noticable just looking through a telescope! Given values for Jupiter's mass ($M_{jup} = 1.9 \times 10^{27} kg$) and radius ($R_{jup} = 70,000\, km$), what do you think its spin angular velocity should be? What is the corresponding period of revolution (i.e., the duration of a Jovian day)? This last can be estimated by watching observable surface features (such as the famous "great red spot") move across the surface. Your teacher will tell you the rotational period that comes from such observations so you can check the accuracy of your prediction. (By the way: do you understand how Jupiter's mass and radius can be known?)

5.5 The Sun's mass is $M_{sun} = 2 \times 10^{30} kg$ and its radius is $R_{sun} = 7 \times 10^{8}\, m$. Observation of Sun Spots progressing slowly and systematically across the visible face of the Sun suggest that the Sun rotates with a period of about 25 days. What do you predict should be height of the Sun's Equatorial bulge and/or its flattening parameter? Should the Sun's oblateness be obvious through a telescope the way Jupiter's is?

5.6 In the discussion of the Earth-Moon tidal interaction, we mentioned that the Moon's orbital angular momentum is proportional to the square root of its orbital radius. Show that this is right.

5.7 Using angular momentum conservation, find the angular velocity at which the Earth and Moon will both move, long in the future when they are finally tidally-locked, face to face. How long will the Earth day be then? How long with the "month" (the period of the Moon's orbit) be?

5.8 Approximate the torque exerted on the Earth by the Moon due to the Earth's tidal bulge. Roughly how long will it take for the Earth and Moon to become tidally locked?

5.9 Use the model developed in Project 5.3 and Equations 5.52-5.53 to calculate the *torque* exerted on the (bulging) Earth by the Moon, say during the part of the monthly cycle when the Earth's spin axis is tilted maximally toward the Moon – i.e., the Lunar equivalent of the Summer Solstice. You should be able to get out an actual honest-to-god number (of Newton-meters or whatever your favorite unit of torque is). Now think about how this torque varies during the monthly Lunar cycle. What do you think the *average* torque is? Now do all of this again for the torque exerted by the Sun (to whatever extent, that is, doing anything again is required). Add the two results together to find the total time-average torque exerted on the Earth. And finally plug the result into Equation 4.99 from Chapter 4 to *predict* the Earth's precessional period. (Recall from Chapter 1 that the actual period is about $26,000$ years. You should get something in this ballpark, which is pretty cool given the crudeness of this model for the Earth. We'll count it as definitely understanding the cause of the observed rate for the "precession of the equinoxes.")

5.10 According to a speculative theory going back to George Howard Darwin (son of the biologist Charles Darwin), our moon might have been formed from material of the Earth's crust flung off by the rotating Earth. How fast would the Earth have had to rotate at that time to make the latter picture plausible?

5.11 Here's a cute little model that will help you understand the tides: consider a little "barbell" type thing made of two masses m connected by a spring of spring constant k and rest length L. Suppose this object is orbiting another object (a star or planet or whatever) of mass M, with an orbital radius R. Consider the various ways it could orbit (axis-on, side-on, no spin angular momentum, spinning fast, etc.) and address, for the different types of orbit (or different moments during the orbit as appropriate): what is the separation between the two masses?

5.12 The moon, like the Earth, is not a perfect sphere. Its biggest "radius" exceeds its smallest "radius" by about 2.2 kilometers. Can you understand this number based on the physics in this chapter? In particular: is the Moon's 2.2 kilometer bulginess a result of rotation or tidal forces or what? What about the fact that the Moon doesn't rotate – i.e., that it always presents the same face to the Earth?

5.13 In the discussion of the torque exerted by the Moon on the Earth's tidal bulges (and its effects) it was mentioned that the length of the day is increasing by 1.6 milli-seconds per century, and that the radius of the Moon's orbit is increasing by 3.5 cm per year. From each of these numbers, calculate the rate of change of the associated angular momentum. They should be about the same (with one positive and one negative), in light of angular momentum conservation for the combined Earth-Moon system. Are they?

5.14 Not long after Pluto's discovery in 1930, its distance from the Sun was measured to be about 39.5 AU. (Actually, Pluto's orbit is highly elliptical, so that's just an average. You should be able to explain how this distance could be measured!) As seen from Earth, around the time Pluto's distance from the Earth is 38.5 AU, its moon Charon appears to oscillate back and forth about Pluto with a period $T = 6.39\,\text{days}$ and with an amplitude of $3.4 \times 10^{-}6$ radians. (This is equivalent to the angular diameter of Charon's orbit being 1.4 arc-seconds.) What is Pluto's mass? What not-quite-true assumption explains why your answer is slightly different than the currently accepted value of $1.52 \times 10^{21}\,\text{kg}$?

5.15 Our derivation of the "Roche limit" for tidal disruption of a moon left something important out. (Actually it left several important things out, but this is the biggest and easiest to address.) For any moon which is in danger of approaching the Roche limit, it is likely that the tidal forces are already strong enough to have gotten the moon into a tidally locked synchronous rotation in which it always presents the same face to the planet. This means, as viewed from an inertial frame, that the moon will be *rotating*, which means that there will be a "centrifugal" tendency for the moon to come apart, in addition to the tidal effect noted in the earlier derivation. It turns out that the centrifugal effect is just about as big as the tidal effect, so it really should be included. So include it! For definiteness: calculate the centrifugal component to the effective gravity at the surface of the moon. This is given by

$$g_c = \omega^2 R_{moon} \tag{5.92}$$

so your only task here is to find an expression for the rotational angular velocity ω in terms of the mass of (M_{planet}) and distance to (r) the planet. Hint: for a tidally locked synchronous orbit the *rotational* and *orbital* angular velocities are the same. Now that you've got that worked out, develop a new-and-improved formula for the Roche limit.

5.16 Using the new and improved formula for the Roche limit that you got from the previous Project, calculate the Roche limit for Saturn. Of course, if you were to look up the fact that Saturn's radius is 6×10^7 meters, you could calculate the Roche limit in meters. But it is more revealing to just calculate the dimensionless multiplier by which the critical distance exceeds the planet's radius. Saturn's average mass density is about $0.7\,\text{g/cm}^3$. What's a reasonable value to use for the mass density of the (perhaps shredded) moon? Is the result more or less consistent with the picture of Saturn in Figure 5.15 and the hypothesis that the rings exist as rings because they are inside the Roche limit?

5.17 You might think some special mechanism is needed to explain how a neutron star could come to be rotating up to a thousand times per second. But in fact, the conservation of angular momentum is sufficient. First, explain qualitatively why the angular momentum of the progenitor star's core should be conserved during the core-collapse supernova which produces the neutron star, and why this collapse would magnify any small initial angular velocity into a much larger angular velocity. Write an expression for the core's moment of inertia in terms of its mass and radius, and then use conservation of angular momentum to derive an expression for the final rotation period as a function of the initial rotation period and the initial and final radii. What initial rotation period is needed to produce a millisecond pulsar? Is this reasonable? (Hint: recall Galileo's sunspot observations from Chapter 2.)

5.18 Estimate the amount by which the gravitational binding self-energy of the core changes when a star undergoes a core-collapse supernova. Work out the actual number in Joules. Is this an increase or a decrease in its energy? Where do you think the missing (or extra?) energy comes from (or goes?)?

5.19 It turns out that only about a hundredth of a percent (0.0001) of the energy difference calculated in the previous Project is converted into visible radiation. (Most of the energy escapes in the form of neutrinos, a type of particle that is copiously produced as a by-product of the electron + proton → neutron reaction which occurs during the collapse.) But still, this is a tiny fraction of a huge amount of energy. Calculate the luminosity of a supernova if this entire energy is given off over a period of about a month (which is about the period during which a typical supernova is at its brightest). For comparison, the Sun's luminosity is about

$$L_{sun} = 4 \times 10^{26}\, W. \tag{5.93}$$

5.20 The text discusses how a neutron star is formed during – and then left behind by – a core-collapse supernova explosion. It is possible, however, for the core to turn not into a neutron star but something else instead: a black hole. As a preliminary definition, we would say that a black hole is any object for which the escape velocity from the surface exceeds the speed of light ($c = 3 \times 10^8\, m/s$). The idea is then that even light cannot escape from the surface, and so the object will appear black. It turns out, however, that according to general relativity, one cannot have such a small, rigid black hole with a well-defined (albeit unobservable) surface. Rather, such an object would necessarily be unstable and collapse indefinitely, forming a point or "singularity." (Actually, this shouldn't be taken too seriously either, since at some point such a high density will be reached that even general relativity doesn't apply – but then nobody has any way to guess what might happen.) In any case, although it doesn't exactly have a "surface," even a point mass will have some specific distance away from it – the so-called "Event Horizon" radius – at which the escape velocity equals the speed of light. The idea is then that light (or anything else, since it is another principle of relativity theory that nothing can go faster than light!) which finds itself inside the Event Horizon can never escape. Find an expression for the Event Horizon radius in terms of the mass of the central body. How big is it (in kilometers) for the Earth? For the Sun? By what factor would you have to compress a neutron star (whose mass is the same as the Sun's mass and whose radius is 10 km) to convert it into a black hole?

5.21 Your teacher will give you some data for the radial velocities of the two stars in an eclipsing binary star system, over time. Determine the masses of the two stars.

5.22 Use the graph and associated data discussed in the text, to work through the calculation of 51 Pegasi b's orbital radius and mass. (Hint: this is mostly an exercise in appropriately simplifying Equations 5.75 and 5.76 for the special case that one mass is much greater than the other. If you do that first, it should then be relatively straightforward to plug in the numbers given in the text for the period and amplitude of 51 Pegasi's radial velocity oscillation.

5.23 Your teacher will give you a file containing data for the radial velocity of a star at a number of times over the course of several months. The intrinsic luminosity of this star is about 10 times the luminosity of the Sun. Try to find a good sinusoidal fit to the data, and use the results to argue for the existence of (and calculate a lower limit on the mass of) an associated extra-solar-planet.

5.24 Stars on the (apparent) fringes of our own galaxy, the Milky Way, appear to orbit the center with a speed of about 225 km/sec. For a star whose galactic radius is 17 kiloparsecs (about twice as far from the galactic center as the Sun), what does this imply about the total mass of the Milky Way galaxy? How does this compare to the results of statistical studies which show that the Milky Way contains about 50 Billion ($10^1$1) stars comparable to the Sun? Based on the numbers given here, what fraction of the Milky Way's total mass is dark matter?