Further Integration by Parts

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Definition

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- Q: Can we use any antiderivative for v?
- A: YES!!

If $dv = g'(x)\,dx$ then the general solution is $v = g(x) + C$, where C is a constant. Consequently the antiderivatives of dv differ by at most a constant.

General antiderivative for v

Consider Equation 1 where we substitute v for $v + C$, for some constant C.
Thus

$$\begin{aligned}
\int u dv &= u(v+C) - \int (v+C)\, du \\
&= uv + Cu - C\int du - \int v\, du \\
&= uv + Cu - Cu + K - \int v\, du \\
&= uv - \int v\, du + K,
\end{aligned}$$

for another contant K.

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for another contant K.
However, we can ignore this contant K and include it as part of the second integral in Equation 1.
Hence Equation 1 works for any choice of antderivative for v!

Further Calculus

Aside:

- Calculus I: "Introduction" to Calculus ideas.
- Calculus II: Develop these Calculus ideas.
- Calculus III: Develop these ideas with vectors!!

Previous students of Calculus 3:
Who dance?!

Reduction Formulas

Sometimes you will find that after you integrate by parts you have an almost identical integral, but one which differs from the original by some parameter, or some constant.

Example

$$\int x^5 e^x \, dx = x^5 e^5 - \int x^4 e^x dx.$$

Notice that the x^5 in the integration has become x^4.

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In general, we let $I_n = \int x^n e^x \, dx$. Then

$$\begin{aligned} \int x^n e^x \, dx &= x^n e^x - n \int x^{n-1} e^x \, dx \\ I_n &= x^n e^x - n \cdot I_{n-1} \end{aligned}$$

, where we can easily see that $I_0 = e^x + C$, for some constant C.

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, where we can easily see that $I_0 = e^x + C$, for some constant C.
So

$$I_5 = (x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)e^x + C$$

More reduction formulas

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Using integration by parts

$$\int e^x \sin(3x)\,dx \quad = \quad e^x\left(-\frac{1}{3}\cos(3x)\right) - \int e^x\left(-\frac{1}{3}\cos(3x)\right)\,dx$$

. We haven't got an integral that looks similar to the original integration, and so we do the process again.

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Thus

$$\begin{aligned} \int e^x \sin(3x)\, dx &= e^x \left(-\frac{1}{3}\cos(3x)\right) - (-\frac{1}{3}) \int e^x \cos(3x)\, dx \qquad (2) \\ &= -\frac{1}{3} e^x \cos(3x) + \frac{1}{9} e^x \sin(3x) - \frac{1}{9} \int e^x \sin(3x)\, dx. \end{aligned}$$

The last integral looks exactly the same as the original one.

Continued...

Let

$$I = \int e^x \sin(3x)\,dx.$$

Then by Equation 2:

$$I = -\frac{1}{3}e^x\cos(3x) + \frac{1}{9}e^x\sin(3x) - \frac{1}{9}I.$$

We now solve for I as follows:-

$$I = \frac{9}{10}\left(-\frac{1}{3}e^x\cos(3x) + \frac{1}{9}e^x\sin(3x)\right).$$

To Try

Visual Calculus: Integration by Parts
Visual Calculus: Java drill problems on Reduction Formulas
Another proof that 1=0