Complex Variables

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Real Life Applications

Real Life quantities that are naturally described through complex numbers

- State of a circuit element descibed by the real numbers voltage V and the current I through it.
  - Can regard as$z = V + iI$.
- A circuit element also may possess a capacitance C and an inductance L.
  - Can regard as $w = C + iL$.
- Complex multiplication:
  - Liken to a light beam passing through a medium that reduces both the intensity and shift the phase.
- Important application:
  - Solving differential equations in their most general form!!

Equations

- Recall that if $z = a + ib$, then$|z| = z \cdot \overline{z} = a^2 + b^2$.
- Example:

Equation of a circle with radius 5 with centre (3,-2).
Satisfies the equation $|z - (3 - 2i)| = 5$.
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So $|x - 3 + (y - 2)i| = 5$ (and its conjugate $|x - 3 - (y - 2)i| = 5$).
Thus

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$$(x - 3)^2 + (y - 2)^2 \quad = \quad 25.$$

Another example

Example: $\Re(z) + 4 = |z - 2|$. Again let $z = x + iy$. So
$x + 4 = |x - 2 + iy|$.
Note $x + 4 = |x - 2 - iy|$. Hence

$$(x-2)^2 + y^2 = (x+4)^2.$$

Solving gives $x = \frac{y^2-12}{12}$.

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Describe or solve the following:

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Solutions:

1. $y = 1/2$
2. $y = \frac{(x^2+9)}{6}$.

Arithmetic and the complex plane

Q: What do simple complex functions do to the complex plane?
A: Transformations

- http://www.youtube.com/watch?v=xuO5lOhXBvM

Triangle Inequality

Let z_1 and z_2 be two complex numbers. Then

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

Geometry of Complex Numbers:
http://math.fullerton.edu/mathews/n2003/ComplexGeometryMod.html

Complex exponential

Compare and contrast the properties of the real exponential function and the complex exponential function.

- Link

Definition:
If $z = x + iy$, then e^z is defined to be the complex number

$$e^z := e^x(\cos y + i \sin y).$$

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Motivation in terms of differntial equations. Using the above formula, we can easily show *De Moivre's formula*

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta).$$

Simple?! Graphs

The graph of $f(z) = e^z$.

- YouTube http://www.youtube.com/watch?v=6dcghhQVvNI.

Studying the repeated application of the function $f(z) = z^2 + c$, for different complex values c, to points in the complex plane.

- YouTube: Fractals;
  http://www.youtube.com/watch?v=BngZUXqKeSk