Complex Variables

2nd February 2010

e^{iy}?

Recall:

- We know what e^x means when x is a real number.
- What does e^{iy} mean?

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Properties:

- $\frac{d}{dz}(e^z) = e^z$; follows by differentiation by substitution;
- $\frac{d}{dy}(e^{iy}) = ie^{iy}$;
- $\frac{d}{d(iy)}(e^{iy}) = e^{iy}$; follows by differentiation by substitution;
- $\frac{d^2}{dy^2}(e^{iy}) = -e^{iy}$;
- $e^0 = 1$.

Differential equations

Let $g(y) = e^{iy}$. We will use differential equations to give another form to $g(y)$.
Have that:

$$\begin{aligned} g(0) &= 1; \\ g'(0) &= i; \\ g''(y) &= -g(y). \end{aligned}$$

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The last equation has a general solution of the form:

$$g(y) = A\sin(y) + B\cos(y).$$

e^z

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$g(y) = \cos(y) + i\sin(y)$.
Hence, if we let $z = x + iy$, then

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x\left(\cos(y) + i\sin(y)\right) \\ &= e^x\cos(y) + ie^x\sin(y) \end{aligned}$$

What does the graph look like?

We can visualize things in 4D. However, here is a projection.
Complex exponential graph

Does the definition work?

Consider the following:

$$\begin{aligned}
e^{iy_1} \cdot e^{iy_2} &= (\cos(y_1) + i\sin(y_1)) \cdot (\cos(y_2) + i\sin(y_2)) \\
&= (\cos(y_1)) \cdot (\cos(y_2)) - (\sin(y_1)) \cdot (\sin(y_2)) \\
&\quad + i\left((\sin(y_1)) \cdot (\cos(y_1)) + (\sin(y_1)) \cdot (\cos(y_2))\right) \\
&= \cos(y_1 + y_2) + i\sin(y_1 + y_2) \\
&= e^{i(y_1+y_2)},
\end{aligned}$$

by the Trigonometric angle sum identities.
Wikipedia explains the angle sum indentities here.

Complex definition

Clearly:

$$e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$$

Consequently, if we let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}.$$

Properties for division of complex numbers follows in a similar way.

Trigonometry

Recall that $z = re^{i\Theta}$, where $r = |z|$ and $\Theta = \arg(z)$.
Thus

$$\begin{aligned}\cos(\Theta) &= \Re(e^{i\Theta}) \\ &= \frac{e^{i\Theta} + e^{-i\Theta}}{2}\end{aligned}$$

and

$$\begin{aligned}\sin(\Theta) &= \Im(e^{i\Theta}) \\ &= \frac{e^{i\Theta} - e^{-i\Theta}}{2i}.\end{aligned}$$

De Moivre's Formula

Theorem

De Moivre's Formula:

$$(\cos(\Theta) + i\sin(\Theta))^n = \cos(n\Theta) + i\sin(n\Theta), \qquad (1)$$

for all $n = 1, 2, 3, \ldots$.

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Note that integrating powers of sine and powers of consine is hard, unless using integration by substitution. However, inetgrating cosine and sine functions is easy!

To try

Evaluate

- $(3(\cos 40^\circ + i \sin 40^\circ))(4(\cos 80^\circ + i \sin 80^\circ))$;
- $\frac{(2(\cos 15^\circ + i \sin 15^\circ))^7}{4(\cos 45^\circ + i \sin 45^\circ)^3}$;
- $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$.

Powers & Roots

Let $z = x + iy$. Then $z^n = (x + iy)^n$.
We can expand this or use De Moivre's formula, Equation 1.
Now let $z = re^{i\Theta}$. Thus $z^n = re^{in\Theta}$.

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Now let $z = re^{i\Theta}$. Thus $z^n = re^{in\Theta}$.
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So $\zeta = \sqrt[m]{r} e^{\frac{i\Theta}{m}}$.
Q: What about the other distinct roots?
A: Utilize trigonometry!

Fact

m distinct roots of unity given by $1^{\frac{1}{m}}$:

$$1^{\frac{1}{m}} = e^{\frac{i2\pi k}{m}} = \cos\left(\frac{2\pi k}{m}\right) + i \sin\left(\frac{2\pi k}{m}\right),$$

for all $k = 0, 1, 2, \ldots, m - 1$.
Complex roots of unity link