On Uncertainty: Some Notes on Probability*

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October 5, 2009

1 Introduction

Themes so far in the class have included certainty and persuasion. In this section of the class we'll study some probability theory and how this can be used, and misused, in arguments. The first job is to understand the basics of probability; these notes are written to try and achieve this.

Going back to the origins of this course, and also the origins of probability theory, we consider the mathematics of dice behaviour. What do we expect to happen? Can we judge how likely a particular outcome is? What strategies arise from dice-based play? How do we evaluate these strategies? We'll consider these questions in the particular context of the game Pig (described in Secton 3). Of course, the lessons learnt have far wider application. In particular, I hope the tools we develop will help when designing your own games, when evaluating arguments that use probability and, of course, when indulging in the noblest of mathematical pursuits: scamming money out of those less numerate than you.

*These notes are adapted from a similar set written for the Games Theorists Play tutorial last semester. Those original notes are more concerned with dice in games and include some more theory and examples in the context of the game Heroscape. I'm happy to share a copy of the original notes with anyone who is interested.

I'll soft-pedal the formal proofs in favour of a practical understanding: while a more formal approach is valuable for many reasons the principle goal for us is to play with the concepts. Before getting stuck into Pig, let's look at five fundamental rules of probability from which the rest of our discussion will flow.

2 Five basic rules

An *event* is a precisely defined potential outcome in a given situation. Examples include getting a 6 when rolling a single standard die, rolling a double on a pair of dice and getting at least three heads when tossing ten coins at once. We talk of the "probabilities" of events; the five rules capture what it is we mean by this. We denote the probability of an event X by $P(X)$.

Rule 1 *For any event X we have $0 \leq P(X) \leq 1$. If $P(X) = 0$ then the event is* impossible*; if $P(X) = 1$ then the event is* certain*.*

This rule simply says that probability operates on a scale from zero to one, with zero probability meaning that the event cannot happen and a probability of one meaning that the event is certain.

Example 1 *Roll a standard die and let X be the outcome. Then we have $P(X = 7) = 0$ and $P(X \leq 6) = 1$. Nothing profound here: it's impossible to roll a 7 on regular die; it's certain that you'll roll at most a 6. You knew this already, but it's good to see the notation in action in familiar situations.*

Our next rule gives meaning to probabilities that lie between zero and one.

Rule 2 *If there are n equally likely outcomes then the probability of any one outcome is $\frac{1}{n}$.*

Example 2 *Roll a standard die and let X be the outcome. Also, abbreviate $P(X = 1)$ by $P(1)$ and so on. We have*

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}.$$

Similarly,

$$P(\text{X is even}) = P(\text{X is odd}) = \frac{1}{2}.$$

The "equally likely" condition is important here. When rolling two dice the probability of a total of 3 is different to that of rolling a total of 7. Rules 4 and 5 will lead to an understanding of this situation.

Rule 3 says "something happens":

Rule 3 *The sum of the probabilities all of the different possible basic outcomes is 1.*

Example 3 *Roll that standard die again. We have:*

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1.$$

Rule 3 suggests that we can add up probabilities. For example, let E be the event that we roll at least one 6 when we roll two dice in turn. We might hope that:

$$P(E) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

Taking this hope further, let F be the event that we roll at least one 6 when we roll seven dice in turn.

$$P(F) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{7}{6} > 1.$$

Oops. What went wrong? The problem is that we gave undue weight to the situation in which more than one die comes up with a 6. The probability

Table 1: Outcomes when rolling two dice

	1	2	3	4	5	6
1	1,1	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

$P(E)$ that we get a six at least once on two dice needs to be smaller: consulting Table 1 we see that $P(E) = \frac{11}{36}$. Only 11 of the "basic outcomes" include a 6, not the 12 we'd expect from our naïve hope.

However, all is not lost. From this unsuccessful addition-of-probabilities experiment we can salvage Rule 4 that tells when we can add probabilities. Call two events *mutually exclusive* if they cannot happen simultaneously. So, rolling a 3 and rolling a 4 are mutually exclusive events when rolling a standard die, whereas when rolling two dice in turn rolling a 6 on the first and rolling a 6 on the second are not mutually exclusive as both can happen.

Rule 4 *If X and Y are mutually exclusive events then $P(X \text{ or } Y) = P(X) + P(Y)$.*

Example 4 *Let's roll two dice again with Rule 4 to hand. Again, let E be the event that we roll at least one six. We can divide E into three mutually exclusive possibilities:*

- *A: we roll a 6 on the first die but not the second,*
- *B: we roll a 6 on the second die but not the first,*
- *C: we roll a 6 on both dice.*

Table 1 tells us the probability of each of these events. $P(A) = \frac{5}{36}$, $P(B) = \frac{5}{36}$ *and* $P(C) = \frac{1}{36}$ *and Rule 4 then gives*

$$P(E) = P(A) + P(B) + P(C) = \frac{5}{36} + \frac{5}{36} + \frac{1}{36} = \frac{11}{36}$$

which we know to be the correct answer.

Fired up with enthusiasm, one may now consider $P(F)$ as defined above: what is the probability that we roll at least one 6 when we roll seven dice in turn. More general questions are now also answerable: what is the probability that we roll at least four 6s when we roll ten dice? At least fourteen 6s when we roll twenty-three dice? And so on. While these are theoretically within reach with our current toolkit, some more results will make the job much easier.

Before Rule 5, here is an immediate consequence of the rules we've seen so far, where "not X" is the event that X does not happen:

Corollary 1 *For any event* X*, we have* $P(X) + P(\text{not } X) = 1$.

This corollary will frequently be useful.

Example 5 *Corollary 1 gives us a more efficient way to solve the problem of Example 4. Let* D *be "not* E*": the event that we do not roll at least one 6. Put differently,* D *is the event that we roll no sixes. Now* $P(D)$ *can be read from the table as* $\frac{25}{36}$ *and we have*

$$P(E) = 1 - P(D) = 1 - \frac{25}{36} = \frac{11}{36}$$

as required.

Maybe there was not such a saving in effort with this example, but in conjunction with Rule 5 this technique will bring many more problems within range.

One more definition before Rule 5. Two events X and Y are *independent* if whether or not X occurs has no bearing on $P(Y)$. Whether a 6 is rolled on each of two dice are independent events. The probability of a 6 on the second die is $\frac{1}{6}$ regardless of the outcome on the first die.[1]

Rule 5 *If X and Y are independent then $P(X \text{ and } Y) = P(X) \times P(Y)$.*

For the final time, let's look at the probability of at least one 6 during successive die rolls.

Example 6 *Rule 5 allows us to perform these calculations without recourse to the tabulation of all of the outcomes when two dice are rolled. This holds out (justified) hope that we can move towards more complex situations such as the seven dice example. Consider D as defined above. Knowing that $P(\text{not } 6) = 1 - \frac{1}{6} = \frac{5}{6}$ when rolling a single die, we calculate that*

$$P(D) = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$$

and may proceed as in the previous example.

Example 7 *Now let's return to $P(F)$. What is the probability that at least one 6 is rolled when seven dice are rolled in turn? While it is possible to enumerate the possibilities and calculate their various probabilities, we more efficiently use the technique of the last example. Let G be the probability that we roll no 6s on the seven dice. Rule 5 gives*

$$P(G) = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = 0.279$$

and now $P(F) = 1 - P(G) = 0.721$ by Corollary 1. In other words, there is about a 72% chance of rolling at least one 6 when rolling seven dice.

[1]Note: this is a departure from the usual presentation of probability theory. Most authors present the more fundamental notion of "conditional probability" and work up to independence as a consequence.

That completes our grounding in probability. Not all questions are yet answerable (or, at least, easily answerable): the probability that we roll at least four 6s when we roll ten dice, for example, needs a little more theory to be tractable.

Before continuing to the particular setting of Pig, here are a few exercises for you to practise the techniques of this section.

Exercises

1. What is the probability of rolling a number less than 5 on a single die?

2. Roll two dice. What is the probability of each of the following events?

a. Rolling a total of exactly 5?
b. Rolling a total of at most 5?
c. Rolling a double?
d. Rolling exactly one 4?
e. Rolling at least one 4?
f. Rolling at least one 4 or at least one 5?

3. What is the probability of rolling at least one 6 when you roll four dice? When rolling ten dice? How many dice do you need to roll to have at least a 90% chance of rolling a 6? A 99% chance?

4. In one of the earliest published works on the theory of probability Galileo investigated a claim common among gamblers of the time that when rolling three dice a total of 10 arose more commonly than a total of 9. Were the gamblers correct? [1]

3 Pig

Before Pig here's a much simpler game. Honest Bob tosses a coin. If it's a head you win \$10, if it's a tail you lose. Honest Bob is charging \$8 to play, do you take him up on the offer? How about if he was charging \$2? \$5? If you answered "no", "yes", "maybe" (in that order) then you already have an intuitive grasp of "expected values".

Let $X_1, X_2, \ldots, X_n$ be the n possible (mutually exclusive) outcomes in some situation. Let $v_1, v_2, \ldots, v_n$ be the values that each of the outcomes is worth to us respectively. We then define the *expected value* of the game, $E(G)$, by

$$E(G) = v_1 P(X_1) + v_2 P(X_2) + \cdots + v_n P(X_n).$$

In words, we've multiplied the probability of each event by its value and added it all up.

Let's go back to Honest Bob to get a sense of what this expected value thingy does. Let X_1 be heads and X_2 be tails. Then, in the first example, $v_1 = 2$ because if heads is the outcome we win \$2 (\$10 winnings minus the \$8 to play). Similarly, $v_2 = -8$. Now,

$$E(G) = (2 \times 0.5) + (-8 \times 0.5) = 1 - 4 = -3$$

(the 0.5s are the probabilities of tossing a head or a tail). An expected value of $-\$3$ means that, over the long term, if you keep playing the game, you'll lose about \$3 per game on average. Of course, there is no one game in which you lose exactly \$3: sometimes you'll come \$2 to the good; other times you'll lose \$8. However, if you play a bajillion times you can expect to be about three bajillion dollars down when you're done.

Running the same analysis on the other two versions of the game, we find that the expected value is \$3 for the \$2-to-play game and \$0 for the \$5-to-play game. So, at \$2-to-play we make a profit in the long-term. At \$5-to-play we break even. Expected values give us a sense of the best choice to make if we are in the situation of making the same choice again and again. Often, the same choice is the best in the instance when you just make that choice once. Given one go at Honest Bob's game, the "no", "yes", "maybe" approach to the three options dictated by the expected value calculations is probably the way to go for individual games too.

The tension between expected values that tell you the best choice with respect to return in the long-run and more immediate concerns is—in a theory of mine formed during the tutorial last semester that inspired this class—a crucial aspect of generating meaningful play in dice games.

So, our rule-of-thumb will be that a positive expected value is good and a negative expected value is bad. In situations with more options, the higher the expected value the better. With that in mind, let's turn to Pig. Quoting Salen and Zimmerman [3, p. 182] quoting Knizia [2], here are the rules of Pig:

> *Object:* The aim of the game is to avoid rolling 1s and to be the first player who reaches 100 points or more.
>
> *Play:* One player begins, then play progresses clockwise. On your turn, throw the die:
>
> · if you roll a 1, you lose a turn and do not score.
>
> · if you roll any other number, you receive the corresponding points.
>
> As long as you receive points you can throw again, and again. Announce your accumulated points so that everyone can easily follow your turn. You may throw as often as you wish. Your turn ends in one of two ways:
>
> · If you decide to finish your turn before you roll a 1, score your accumulated points on the notepad. These points are now safe for the rest of the game.
>
> · If you roll a 1, you lose your turn and your accumulated points.
>
> Record all scores on the notepad and keep running totals for each player. The first player to reach 100 points or more is the winner.

Will expected value considerations let us recreate the Pig strategy that Salen and Zimmerman [3, p. 183] report that Knizia calculates as best for Pig? That is, stop rolling once you've amassed 20 or more points in a turn.

At any point within a turn there are two options: roll or do not roll ("stick"). We'll denote them R and S respectively. Suppose you have k points at this point. We'll write $E(R|k)$ for the expected value of strategy R given that we have k points and $E(S|k)$ for the non-rolling analogue.

Example 8 *Suppose we have* 12 *points so far this turn and we must choose whether to roll or stick. What are the expected values for each strategy? Sticking is easy to calculate. If we stick then we bank our points:* $E(S|12) = 12$. *Now suppose that we roll. The probability of a* 1 *is* $\frac{1}{6}$ *and in this case we lose our* 12 *points, a value of* 0. *The probability of a* 2 *is* $\frac{1}{6}$ *and this gives us 14 points; the probability of a* 3 *is* $\frac{1}{6}$ *and this moves us to 15; and so on.*

$$E(R|12) = (0\times\frac{1}{6})+(14\times\frac{1}{6})+(15\times\frac{1}{6})+(16\times\frac{1}{6})+(17\times\frac{1}{6})+(18\times\frac{1}{6}) = 13.333.$$

As $13.333 > 12$ *our expected value is maximised if we roll again.*

Example 9 *Suppose now that we have* 30 *points on the turn.* $E(S|30) = 30$ *and running the numbers as before we find that* $E(R|30) = 27.5$. *In this case we are better off sticking.*

We could continue in the vein of these two examples and zero in on the precise number at which the cut-off between rolling and sticking occurs. However, we can be a little more efficient. Let c be that cut-off number. When we have c points during a turn the expected value will be the same whether we roll or stick (why?). That is, $E(R|c) = E(S|c)$. Therefore

$$(0\times\frac{1}{6})+((c+2)\times\frac{1}{6})+((c+3)\times\frac{1}{6})+((c+4)\times\frac{1}{6})+((c+5)\times\frac{1}{6})+((c+6)\times\frac{1}{6}) = c.$$

Simplifying the left hand side, this reduces to

$$\frac{5c+20}{6} = c$$

and hence $c = 20$. Exactly what Knizia prescribes!

Here is where we meet that tension between expected values and best move again. Early on when there are many moves to go and the goal is to get points on the board as quickly and reliably as possible, following the stick-on-20-and-higher rule will be the best strategy. However, if your opponent has 99 maybe it's worth taking more risks; if you're comfortably ahead, maybe taking fewer risks is a better strategy.

Exercises

1. Mathematics for fun and profit. Choose a non-cubic Platonic solid (all of which are commonly used for dice; D4, D8, D12 and D20 are the nerdy names for them). You can play Pig with your choice of die. Perform a similar analysis to that in this section for the "D6"; borrow your chosen die from Eric; win money and/or favours from your friends.

2. Consider these alternative pig dice:
a. A six-sided die with sides $1, 1, 5, 5, 6, 6$,
b. A six-sided die with sides $1, 1, 1, 1, 12, 36$,
c. A twelve-sided die with sides $1, 2, 2, 2, 2, 2, 3, 3, 3, 4, 4, 5$.
For each of these dice analyse the expected values as we did for a regular die. Given the choice against a player using a standard die, which one would you choose?

3. Wild Boar is played as follows: Each player selects a die from the choice of varied dice (one die is a standard six-sider; some have different numbers of sides; they tend to have different values on each side; at least one has a negative number on at least one side). Once each player has a die, they follow the rules of Pig with each player using his/her own die. Design some fair dice to go in the bag. Why are they fair? Is it possible to have a fair die in which one side has the number 100? Which range of dice as the choices do you think will lead to the most meaningful play? [Dice are regularly made in the shape of the Platonic solids, making 4, 6, 8, 12 and 20 siders all common. However, by using "spinners" (or computers) rather than dice, it is easy to have something that acts like a 51-sided die (or any other number). Feel free to experiment with strange numbers of sides in this exercise.]

4. Building on the previous two exercises, design a Pig-like game in which players use different dice at different points of the game.

4 Where to now?

Real life is more complicated than Pig (you may already have realised this). However, I maintain that some of the tools and insights that we now have can be used to better understand real-life issues. Over the next week we'll look at instances of probability used in argument, and at how a better understanding of probability can help us avoid some flaws in reasoning.

References

[1] D. J. Bennett, *Randomness*, Cambridge, Massachusetts: Harvard University Press, 1998

[2] R. Knizia, *Dice Games Properly Explained*, Tadworth, Surrey: Elliot Right Way Books, 2001.

[3] K. Salen and E. Zimmerman, *Rules of Play: Game Design Fundamentals*, Cambridge, Massachusetts: MIT Press, 2004.